Propagation of errors in different phase-shifting algorithms: a special property of the arctangent function

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ABSTRACT

The mathematical properties of the arctangent function are investigated, revealing the mechanism of error propagation in digital phase-shifting applications. The analysis gives a systematic approach to the assessment of errors and, as such, provides a valuable tool for understanding the propagation of errors in any phase-shifting algorithm.

1. INTRODUCTION

High precision phase measurements in both interferometric and non-interferometric structured illumination systems rely heavily on phase shifting techniques [1,2]. Several papers have shown that the generalised phase-shifting algorithm can be derived from the least squares method [3,4]. The calculation of phase in any phase-shifting algorithm involves the ubiquitous arctangent function. Although many papers [5-19] have been published about the effects of various systematic and random errors on the final calculated phase, little, if any, attention has been given to an important and universal property of the arctangent function - namely frequency-shifting. This refers to phase measurement errors having specific frequency components being propagated to the calculated phase at frequencies that differ from those of the source errors. When phase errors are considered in this way, error propagation mechanisms are combined in one systematic process which is easier to understand than the previously reported ad hoc analyses. As a result, sources of error can be readily identified and the susceptibility of different algorithms to these errors can be easily assessed.

2. PHASE CALCULATION EXPRESSIONS

The calculation of the phase in virtually all phase-shifting techniques is derived from the expression:

$$\Phi(x,y) = \arctan\left[\frac{S\,(x,y)}{C\,(x,y)}\right] + 2m\pi \qquad (1)$$

where $\Phi(x,y)$ is the two dimensional phase distribution as a function of the spatial coordinates x and y, S (x,y) and C(x,y) are quadrature functions derived from a number of interferograms (or intensity maps) using a predetermined algorithm. m is an integer chosen to allow the arctangent function to be calculated modulo 2π. Its effect can be ignored in the following analysis.

From equation (1) it follows that:

$$S(x,y) = B(x,y) \sin \Phi(x,y) \tag{2}$$

$$C(x,y) = B(x,y) \cos \Phi(x,y) \tag{3}$$

where B(x,y) is the amplitude of S and C functions.

Errors encountered in phase measurements - for example, phase-shift errors, non-linearities and multiple interference beams - can be represented as a combination of errors in the numerator and denominator of the argument of the arctangent. The trivial effect of phase quantization will be ignored here.
The actual numerator, denominator and phase in the presence of errors can be defined as follows:

$$\left.\begin{array}{ll} \text{actual numerator} & S' = S + \Delta S \\ \text{actual denominator} & C' = C + \Delta C \\ \text{calculated phase} & \Phi' = \Phi + \Delta\Phi \end{array}\right\} , \tag{4}$$

where ΔS , ΔC, and $\Delta\Phi$ are the errors in the numerator, denominator and phase, respectively.

The explicit variation with x and y has been dropped for simplicity. The actual phase calculation is now given by:

$$\Phi' = \arctan\left[\frac{S'}{C'}\right] . \tag{5}$$

Using the following trigonometric identity:

$$\tan(\Phi + \Delta\Phi) \equiv \frac{\tan\Phi + \tan\Delta\Phi}{1 - \tan\Phi \, . \tan\Delta\Phi} , \tag{6}$$

and equations (2), (3) and (5), the phase error can be written as

$$\tan\Delta\Phi = \frac{\Delta S \cos\Phi - \Delta C \sin\Phi}{B + \Delta C \cos\Phi + \Delta S \sin\Phi} . \tag{7}$$

However, it is much more useful to represent the errors in normalised form.

If $e(x,y)$ is the normalised error in the numerator
and $\epsilon(x,y)$ is the normalised error in the denominator,

$$\left.\begin{array}{ll}\text{then} & e = \dfrac{\Delta S}{B} \\ \text{and} & \epsilon = \dfrac{\Delta C}{B}\end{array}\right\} \quad . \tag{8}$$

Consequently, equation (7) can be written as

$$\tan \Delta\Phi = \frac{e \cos \Phi - \epsilon \sin \Phi}{1 + \epsilon \cos \Phi + e \sin \Phi} \quad . \tag{9}$$

A less general form of this equation has appeared in different guises [5,7,8,9,10,16,18,19] for specific errors. However, the universal significance of (9) does not appear to have been noted in any of the above references.

3. APPROXIMATION TO PHASE ERROR CALCULATIONS

Although equation (9) is exact and valid for all types of errors, it is often adequate to use an approximation which has sufficient accuracy for the range of errors encountered. Otherwise, the use of equation (9) is recommended.

The arctangent function can be expanded as a power series:

$$\arctan p = p - \frac{p^3}{3} + \frac{p^5}{5} + 0\ (p^7) \tag{10}$$

for $[p^2 < 1]$, where $0(p^7)$ includes all terms of order seven and above. For example, the first term alone may be used with less than 1% error if p satisfies the following criterion.

$$| \ p \ | \leq 0.17 \ . \tag{11}$$

Or, equivalently:

$$| \arctan p \ | \leq 0.17 \quad (9.7^\circ) \ . \tag{12}$$

The reciprocal of the denominator in (9) can be approximated by the binomial expansion as

$$(1 + e \sin \Phi + \epsilon \cos \Phi)^{-1} = 1 - e \sin \Phi - \epsilon \cos \Phi + 0\ ([\text{error}]^2) \tag{13}$$

where $0([\text{error}]^2)$ includes all second order and higher terms in e and ϵ.

Combining equations (9), (10) and (13) gives:

$$\Delta\Phi = e \cos \Phi - \epsilon \sin \Phi + \frac{(\epsilon^2 - e^2)}{2} \sin 2\Phi - \epsilon e \cos 2\Phi + 0\ ([\text{error}]^3) \quad . \tag{14}$$

Consider the first order approximation, $\delta\Phi_1$

$$\Delta\Phi = \delta\Phi_1 + 0\ ([\text{error}]^2)\ , \tag{15}$$

$$\delta\Phi_1 = e \cos \Phi - \epsilon \sin \Phi\ . \tag{16}$$

The second order approximation $\delta\Phi_2$, can be defined similarly:

$$\Delta\Phi = \delta\Phi_2 + 0\ ([\text{error}]^3)\ , \tag{17}$$

$$\delta\Phi_2 = \delta\Phi_1 + \frac{(\epsilon^2 - e^2)}{2} \sin 2\Phi - \epsilon e \cos 2\Phi\ . \tag{18}$$

Equation (16) shows clearly that the approximate phase error is the sum of the numerator error (e) modulated by cos Φ and the denominator error (ϵ) modulated by sin Φ. An analogy with amplitude modulation in communication theory is apparent. Amplitude modulation results in a carrier frequency shifting a signal frequency into two sidebands; one at the sum frequency, the other at the difference frequency. Here the carrier frequency is represented by Φ and the signal by e and ϵ both of which can be expressed as a Fourier series (see section 4).

To assess the validity of the first order approximation, let D_1 be the difference between the exact phase error and the first order approximation to the phase error. From equations (15) and (18) it follows that

$$D_1 = \Delta\Phi - \delta\Phi_1 = \frac{(\epsilon^2 - e^2)}{2} \sin 2\Phi - \epsilon e \cos 2\Phi + 0\ ([\text{error}]^3) \tag{19}$$

Typically, with 8 bit digitization, it is desirable that the calculation of phase and phase error should be accurate to $\pm \frac{1}{2}$ level in 256 levels, hence

$$|D_1| \leq 2\pi \cdot \frac{1}{2} \cdot \frac{1}{256} \text{ radians}\ . \tag{20}$$

This defines the valid range of approximation for modulo 2π phase measurement. There are 3 extreme cases for which (16) should be valid:

$$\text{(a)}\quad e = 0 \Rightarrow |\ \epsilon^2 \sin 2\Phi\ | \leq \frac{\pi}{128} \Rightarrow |\epsilon| \leq 0.157 \tag{21}$$

$$\text{(b)}\quad \epsilon = 0 \Rightarrow |\ -e^2 \sin 2\Phi\ | \leq \frac{\pi}{128} \Rightarrow |e| \leq 0.157 \tag{22}$$

$$\text{(c)}\quad \epsilon = \pm e \Rightarrow |\ \mp e^2 \cos 2\Phi\ | \leq \frac{\pi}{256} \Rightarrow |e| \leq 0.111 \tag{23}$$

Case (c) is the most stringent requirement. In conclusion, the first order approximation has no significant error if the following conditions are met:

$$|e| \leq 0.111 \tag{24}$$

$$|\epsilon| \leq 0.111 \tag{25}$$

If the errors exceed the above limits, then equation (14) may be more suitable for calculations. Equation (14) clearly shows how inter-modulation and cross-modulation terms are shifted by 2Φ. However these terms are a much smaller than those in $\delta\Phi_1$, and, therefore, the following analysis will consider only the terms which appear in equation (16).

4. FREQUENCY-SHIFTING OF ERRORS

Many papers [2,4,5,8-11,14-16,18] have noted the Φ, 2Φ, 3Φ and 4Φ dependence of phase errors. The explicit $(n+1)\Phi$ dependence has been discussed recently [19]. However, in this same paper, the $(n-1)\Phi$ dependence is overlooked although it is implicit in some of the comprehensively filled tables of phase measurement errors. The generalised $(n+1)\Phi$ and $(n-1)\Phi$ dependence will now be derived. The key to this derivation is in writing both e and ϵ as Fourier series. The following series represent the harmonic components of the normalised errors defined in equation (8).

$$e = e(x,y) = \sum_{n=0}^{N} e_n(x,y)\cos[n\Phi(x,y)+\alpha_n] \tag{26}$$

$$\epsilon = \epsilon(x,y) = \sum_{n=0}^{N} \epsilon_n(x,y)\cos[n\Phi(x,y)+\beta_n] \tag{27}$$

where α_n and β_n are phase offsets.

The values of n correspond to the following error sources:

- $n = 0$ corresponds to errors independent of phase, for example any intensity offset which varies from sample to sample in the phase-shifting process,
- $n = 1$ corresponds to errors directly related to the phase, for example, phase-shift errors,
- $n = 2$ corresponds to errors produced by the detector non-linearity or a second harmonic in intensity,
- $n = 3$ corresponds to errors produced by detector non-linearity of third or higher order, or a third harmonic in intensity.

Equation (16) can now be rewritten:

$$\delta\Phi_1 = \sum_{n=0}^{N} e_n\cos\Phi\,\cos(n\Phi+\alpha_n)$$

$$-\sum_{n=0}^{N} \epsilon_n\sin\Phi\,\cos(n\Phi+\beta_n)\ . \tag{28}$$

Using the standard trigonometric identities in equation (28) results in

$$\delta\Phi_1 = \sum_{n=0}^{N} e_n \{\cos([n+1]\Phi + \alpha_n) + \cos([n-1]\Phi + \alpha_n)\} /2$$

$$- \sum_{n=0}^{N} \epsilon_n \{\sin([n+1]\Phi + \beta_n) - \sin([n-1]\Phi + \beta_n)\} /2 \qquad (29)$$

Equation (29) is an important result since it gives explicitly the sum and difference frequencies of phase errors inherent in the arctangent calculation. Harmonic components of the phase error can now be linked directly to their sources in the numerator and denominator of the arctangent argument. Because of the general nature of this analysis, all the preceding results are valid for any technique which evaluates the phase from an arctangent function. This means that Fourier transform and spatial synchronous techniques are covered as well as the phase-shifting techniques considered here.

It is important to note that an error in the numerator or denominator of the arctangent argument with an harmonic component at $n\Phi$ will result, generally, in an error in the calculated phase with harmonic components at both $(n+1)\Phi$ and $(n-1)\Phi$. However, if e_n and ϵ_n have a certain quadrature relation then <u>either</u> the $(n-1)\Phi$ or the $(n+1)\Phi$ terms can cancel out. This may explain the absence of an explicit $(n-1)\Phi$ in previous literature [19].

When the numerator and denominator errors occur at the fundamental frequency (n=1) the frequency-shifted phase error components occur at the zero and 2Φ harmonics. The zero (or DC) harmonic has previously been considered a mere artifact of the process. In the light of this new analysis the DC term is just another occurrence of the $(n-1)\Phi$ harmonic.

5. APPLICATION OF ARCTANGENT ANALYSIS TO PHASE-SHIFTING ALGORITHMS

The power of this error analysis technique should not be underestimated although the arctangent function evaluation is just one (last) step in the application of a phase-shifting algorithm. The full process is illustrated schematically in figure 1. Not shown are noise sources which could be inserted at any points in the signal flow to represent actual measurement processes. The first step shown is the effect of detector nonlinearity on the input signal $g(x,y,t)$. Here t is a general phase-shift parameter which may be a temporal or spatial quantity [18,20]. Much phase-shifting interferometry today uses CCD detector arrays which are highly linear in the usual operating range. Consequently step 1 may be omitted in many cases. So for linear detectors the signal $\hat{g}$ is just a scaled version of g with scaled harmonic content. Step 2 can be considered as two discrete correlations in which each sample of the signal $\hat{g}$ has weights A_n or B_n. The weights A_n and B_n define the phase-shifting algorithm. This step can be considered - equivalently - as a linear filtering process. Again only harmonic components already present can get through. In fact some harmonics are suppressed and others enhanced[2,5,6,18,19]. For example the common four-sample (90°) algorithm suppresses the second harmonic. Step 2 of figure 1 only

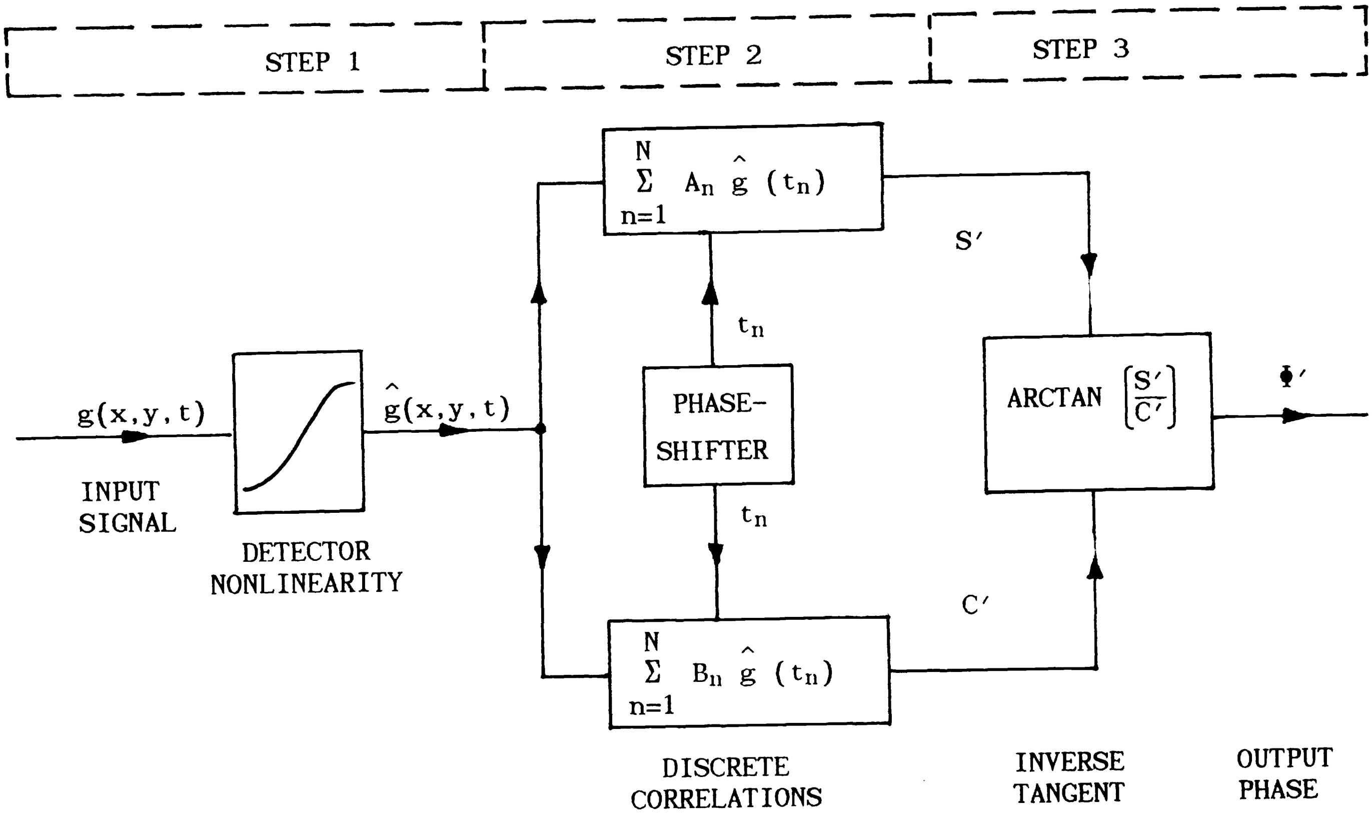

Figure 1. Phase-shifting process - schematic

represents the operation of fixed-step algorithms which account for most algorithms in use. Adaptive-step algorithms - Carre's[21] for example - cannot be represented by a linear filtering process. With some small modifications we can incorporate the Carre algorithm but the resultant nonlinear filter can now generate new harmonic components from its input.

The filter process always occurs before the frequency-shifting arctangent operation. Step 3 of figure 1, is the arctangent evaluation. The crucial point here is that for linear detectors (eg CCD) and fixed-step algorithms it is only the arctangent which generates harmonic components not necessarily present in the input signal g. This harmonic generation process as outlined in section 4 can provide valuable insight into the error propagation process. A full computer simulation on the other hand, although producing exact outputs, nevertheless obscures the error propagation mechanism and does not readily allow the user to identify critical process parameters suitable for optimization (except by trial and error).

The frequency-shifting effect of the arctangent can be utilised for diagnostic purposes. From the harmonic content of the calculated phase inferences can be drawn about the input signal or the algorithm defects. Systems with non-sinusoidal signals such as multiple-beam interferometers or projection moire systems can greatly benefit from such diagnostics. During our experiments with a phase-shifting moire profilometry setup [23], the error-compensating 5-sample algorithm [10] was found to produce a significant amount of phase noise at 4Φ. Fourier analysis of the phase also indicated 2Φ noise but no components at 6Φ. The tentative conclusion was that 3Φ (third harmonic) components must exist in the numerator and/or denominator. Further analysis of the algorithm revealed that a large third harmonic component in the projected grating intensity could be responsible. This was confirmed later by detailed intensity measurements. The frequency-shifting approach simplified the diagnosis in this instance.

Finally, equation (9) reveals that a general condition for zero phase error $\Delta\Phi = 0$ does not require the numerator and denominator errors to be zero but instead $e = E \sin \Phi$ and $\epsilon = E \cos \Phi$ where E can have any value. It is no coincidence that the well known 5-sample (90°) algorithm [10] satisfies these conditions even when phase-shift errors are present. Other algorithms can be designed with this property [20,22].

6. CONCLUSION

A hitherto overlooked property of the arctangent function has been derived and used to clarify details of previous ad hoc analyses. The application to phase-shifting techniques has been outlined in a modular scheme which introduces a diagnostic technique based on harmonic analysis of phase errors.

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