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A Beginner's Guide To The Fractional Fourier Transform
Part 1

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Prologue

Just two years ago the phrase "fractional Fourier transform" would have triggered the head-scratching reflex in 99.9% of opticists. Meanwhile there has been a veritable explosion in the number of papers published on the subject of the fractional Fourier transform (referred to hereafter as the FractFT for want of a more elegant yet compact abbreviation). The majority of these papers have appeared in the leading optics journals of the United States of America and Europe. A quick count through the better known papers gives the following score: 1992 zero, 1993 five, 1994 twenty, 1995 six so far. The pervasive influence of the FractFT has been such that its mention only causes itchy scalp in a mere 99% of opticists today. If this article must state its major objective, then it is to perplex 90% of the readers (and thus inform the remaining 10%, which represents a ten-fold increase in current awareness!)

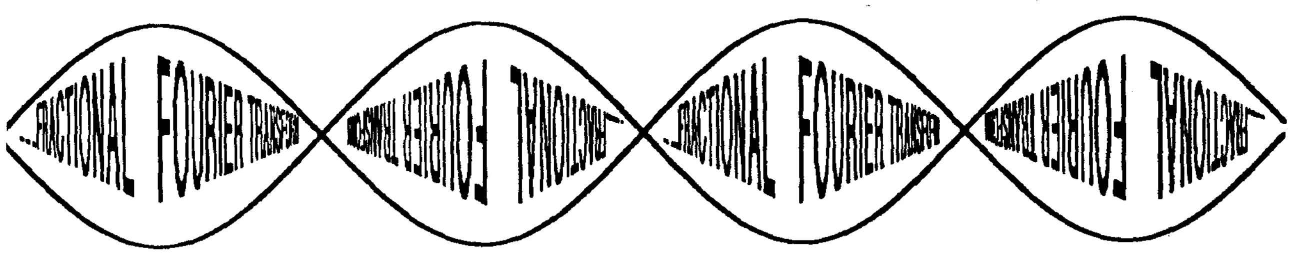
The beginner's guide will be presented in two parts. The first part, in this issue, is concerned mainly with the fascinating history of the FractFT and some speculation about the development of scientific ideas. The second part, to appear in the next issue, aims to make the main ideas behind the FractFT accessible and to discuss the numerous applications in optical systems.

Fourier Transforms & All That . . .

Before going any further it is wise to reveal the subject of our attention. Many readers will be familiar with the common-or-garden Fourier transform and many of its mathematical properties. Unfortunately the FT (as it is affectionately known) often invokes fear and loathing rather than love and understanding. It is necessary to know a few details about the FT before the concept of a FractFTT can have any meaning.

For those with virtually no knowledge of the FT an introductory text on Fourier theory is strongly recommended.[1] The quintessential feature of Fourier analysis is that any function (well *almost* any function, fractals and some other nasties excepted) can be represented as the sum of a carefully chosen selection of sine and cosine functions with a range of frequencies. All that the Fourier transform does is to *calculate* the varying proportions of the sinusoidal terms at each and every frequency. A familiar example of the Fourier transform is the spectrum analyser display found on many hi-fi systems (including the ultra-trendy black display on a matt black background). Such a display approximates the FT by displaying the absolute proportions of harmonic components at a number of predefined frequency bands.

It really is impossible to delay further the introduction of some mathematical definitions.



The first is the definition of the FT as it will be used in all following work

$$G(u) = \int g(x)e^{-2\pi iux}dx$$

For our purposes it is more convenient to represent the above in operator notation

$$G(u) = F_1 g(x) = \int g(x)e^{-2\pi iux}dx$$

Here F_1 represents the FT operator. The function $g(x)$ is the function to be transformed. The kernel in the integrand $e^{-2\pi iux}$ can be thought of as the sine and cosine terms at a particular frequency u. The operator notation allows some important theorems in FT theory to be presented in a particularly simple way. It is well known that the application of the FT twice in succession leads to a simple co-ordinate reflection:

$$F_2 g(x) = F_1 F_1 g(x) = g(-x)$$

Thrice:

$$F_3 g(x) = F_1 F_1 F_1 g(x) = F_1(F_2 g(x)) = F_2(F_1 g(x)) = G(-u)$$

Four times:

$$F_4 g(x) = F_2(F_2 g(x)) = F_2 g(-x) = g(x)$$

So F_N represents N applications of the FT operator (N=integer). Four applications of the FT operator returns a result identical to the initial function. Some perceptive minds have pondered the four unit periodicity of the FT operator and asked the question,
"Is it possible to define an FT operator F_a where a is no longer an integer?". Such an operator is required to retain some of the fundamental properties of the FT operator such as linearity $F_a(g+h) = F_a g + F_a h$

addivity of order $F_a F_b(g) = F_{a+b}(g)$

identity with FT $F_a g = F_1 g$ for $a = 1$

and commutation $F_a F_b g = F_b F_a g$.

These few constraints are necessary to uniquely define a family of transforms now known affectionately as the Fourier transform of fractional order. The above development is rather oblique so that the perspicacious among us may ask, "What has this got to do with optics?"

The answer is, "Plenty", but the details are deferred to the next exciting instalment. In the meantime consider the facts, as they appear at present, relating to the independent discovery (invention is perhaps the more appropriate word) of the FractFT on at least seven separate occasions.

FIRST OCCASION, 1937: A paper published by E.U. Condon[2], then located at Princeton has the promising title, "Immersion of the Fourier tranform in a continuous group of functional transformations". Early on in this predominantly mathematical paper the author observes that "there exists a continuous group of functional transformations containing the ordinary Fourier transforms as a subgroup". The term fractional FT does not appear for another 33 years. Instead the analogy with rotation is used for the fractional index a. Condon defined an index θ which corresponds to $\pi a/2$. The simple case of four successive FTs gives a θ equal to 2π or one complete revolution.

In 1961 there is a reference to Condon's work in a paper by V. Bargman.[3] Again the paper is predominantly mathematical in style (with the emphasis on quantum mechanics of group theory) and the author originates from Princeton.

SECOND OCCASION, 1973: Just 36 years after the first occurrence N.G. de Bruijn[4] publishes a 75 page magnum opus. The paper contains in its first few lines the frank admission that "this section has not been too well organised". Try getting that past a journal editor now! The FractFT appears within the first 10 pages and is referred to in the plural as 'generalized functions'.

THIRD OCCASION, 1980: A period of 7 years has elapsed and Victor Namias coins the phrase "fractional Fourier transform" in a mathematical journal.[5] The application is again quantum mechanics, yet the paper is surprisingly clear and well written. The clarity was apparently due to a lack of mathematical rigour but this was rectified in 1987 by McBride and Kerr.[6] In the years 1987-89 and beyond an Australian based mathematician, David Mustard,[7, 8] developed aspects of the FractFT and refers to the work of Condon, Bargman and Namias.

FOURTH OCCASION, 1982: Up until this date all occurrences of the fractional FT have originated from quantum mechanical considerations. A pair of electrical engineers called Bradley Dickinson and Kenneth Stieglitz[9] wrote a paper on the eigenvectors of the DFT (discrete Fourier transform). Halfway through the idea of a fractional DFT pops up. A number of promising applications in signal processing are proposed but nothing more seems to have been published. Surely it wasn't that bad an idea?

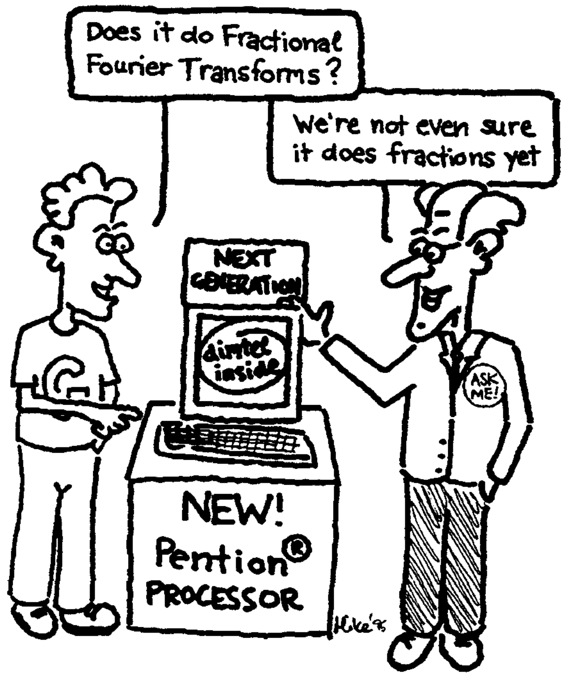
FIFTH OCCASION, 1991: Time for NASA to get in on the act. David Bailey and Paul Swarztrauber[10] notice the simple inclusion of a fractional factor in the exponent of the DFT has some interesting repercussions upon the computational efficiency of many transform based algorithms. A subsequent paper published in 1994[11] is still oblivious to the work of others in this area.

SIXTH OCCASION 1993: The priority in publication is now becoming unclear. Early in the year a conference paper entitled "An introduction to the angular Fourier transform" by Luis Almeida[12] appears. Careful inspection reveals a FractFT by any other name... Applications considered are related to time-frequency representations in signal processing.

SEVENTH OCCASION, 1993: At last! The perfect correspondence of the FractFT with optical propagation in a variety of systems is noticed by David Mendlovic and Haldum Ozaktas while working in Germany.[13, 14] This point marks time zero for the big bang in published work on the application of the FractFT in optical propagation problems. Before publishing, the authors traced the earlier work of Namias. The authors identified the material embodiment of the FractFT in an optical device (or system) known as a gradient index rod lens (see[15] for example). The optical field in any plane perpendicular to the lens axis can be expressed as the FractFT of the field in any other plane. The separation of the planes is directly proportional to the fractional order of the FT. Later publications deal with the FractFT in single lens optical systems.

Epilogue

Fortunately there is little room left for my pontification upon the failure of information technology to make prior art accessible. Instead I present the following plausible conjecture: that it is possible to identify an earlier reference to



the fractional Fourier transform albeit in epic poem form. The earliest reference is none other than "The Hunting of the Snark" by Lewis Carroll[16]. The game is given away in the second verse. Some retrospective modification to this text now needs to be made as follows:

ZEROTH OCCASION, 1874: The first two verses of the "The Hunting of the Snark" are as follows:

> "Just the place for a Snark!", the
> Bellman cried,
> As he landed his crew with care,
> Supporting each man on the top of the
> tide
> By a finger entwined in his hair.
>
> "Just the place for a Snark!, I have said
> twice
> That alone should encourage this crew.
> Just the place for a Snark! I have said it
> thrice.
> What I tell you three times is true".

The last line is obviously a reference to a FractFT of order 4/3 which has a repeat period of 3. The poem is written in eight fits where the word *fit* has at least 3 meanings

i) a convulsion
ii) a canto, or main division of poem
iii) Fractional Integral Transform

The final line of the poem is:

For the Snark was a Boojum, you see.

Lewis Carroll was the well-guarded pen-name of the Rev. Charles Lutwidge Dodgson a lecturer in mathematics, at Oxford. Carroll was noted for his nonsensical sense of humour. The idea of calling the FractFT a Boojum does seem attractive and fitting. Some readers may remain unconvinced by the preceding argument so that, upon reflection, a compromise may be reached and the FractFT called a mere Snark!

Aknowledgements

I would like to thank Prof. Colin Sheppard for the many enlightening discussions preceding this work. David Mustard provided several of the early references included here and answered a number of my questions regarding the FractFT.

NEXT ISSUE: Part 2, the nitty gritty...

References

1 I would be pleased to hear from readers who can suggest a good introductory text, otherwise try the classic text: R. N. Bracewell, The Fourier transform and its applications, McGraw Hill, New York.

2 E. U. Condon, "Immersion of the Fourier tranform in a continuous group of functional transformations," Proc. National Academy of Sciences **23**, 158-164, (1937).

3 V. Bargman, "On a Hilbert space of analytic functions and an associated integral transform," Comm. Pure and Applied Maths **14**, 187-214, (1961).

4 N. G. de Bruijn, "A theory of generalised functions, with applications to Wigner distribution and Weyl correspondence," Nieuw Archief voor Wiskunde **21**, 205-280, (1973).

5 V. Namias, "The fractional Fourier transform and its application to quantum mechanics," J. Inst. Maths Applics **25**, 241-265, (1980)

6 A. C. McBride, and F. H. Kerr, "On Namias's fractional Fourier transform," IMA Journal of Applied Mathematics **39**, 159-175, (1987).

7 D. Mustard, "Lie group imbeddings of the Fourier transform and a new family of uncertainty principles," School of Mathematics Preprint AM87/13, The University of New South Wales, 1987.

8 D. Mustard, "The fractional Fourier transform and the Wigner distribution," School of Mathematics Preprint AM89/6, The University of New South Wales, 1989.

9 B. W. Dickinson, and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," IEEE Transactions on Acoustics, Speech, and Signal Processing **30**, 25-31, (1982).

10 D. H. Bailey, and P. N. Swarztrauber, "The fractional Fourier transform and applications," SIAM Review **33**, 389-404, (1991).

11 D. H. Bailey, and P. N. Swarztrauber, "A fast method for the numerical evaluation of continuous Fourier and Laplace trans forms," SIAM J. Sci. Comput. **15**, 1105-1110, (1994).

12 L. B. Almeida, "An introduction to the angular Fourier transform," IEEE Int. Conf. Acoust., Speech, Signal Processing, (Minneapolis, MN), (1993), 257-260.

13 H. M. Ozaktas, and D. Mendlovic, "Fourier transforms of fractional order and their optical interpretation," Optics Communications **101**, 163-169, (1993).

14 D. Mendlovic, and H. M. Ozaktas, "Fractional Fourier transforms and their optical implementation: I," Journal of the Optical Society of America, A **10**, 1875-1881, (1993).

15 Melles Griot, Optics Guide 5, Melles Griot,

16 M. Gardner, ed., The Annotated Snark. Lewis Carroll, (1962).