A BEGINNER'S GUIDE TO THE FRACTIONAL FOURIER TRANSFORM

Part 2: A Brief History of Time Frequency Distributions

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Earlier this year (Vol. 9, Issue 2) I presented a potted history of the fractional Fourier transform (FractFT from hereon) and promised the "nitty gritty" in Part 2. A lot has happened since then; I've lost count of the new papers appearing with Fractional Fourier Transform in the title. Also my attention has been drawn to additional publications which further complicate the history of the FractFT[17]. Readers interested in specific details of the mathematical analysis are advised to consult the references given at the end of this article. The objective now is to present the main ideas behind the FractFT and discuss some applications.

The Fractional Fourier Transform & Propagation in GRIN Media

The FractFT arises most naturally in the analysis of optical propagation in graded index (GRIN) media. The well known SELFOC lens has an axially symmetric quadratic variation in refractive index. Ray propagation in such a medium can be shown to consist of periodically refocussing sinusoidal ray paths as shown in figure 1. One way of understanding this effect is to consider the GRIN medium as a series of thin lenses which gradually focus a diverging group of rays.

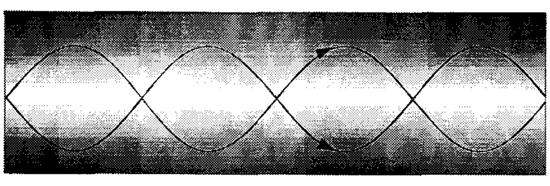

Figure 1 Ray propagation in a GRIN medium

Moving on now to the propagation of an optical field in such a medium... The self-modes of a quadratic GRIN medium are Hermite-Gaussian (H-G) functions. In fact these modes are the same as the familiar modes of a laser resonator. Conveniently, these modes can be shown to form a complete orthogonal set. In other words, any function can be represented as a summation of the H-G functions. Hence any distribution of the input field can be represented using H-G functions#. The propagation of this field can then be reduced to the simpler problem of the propagation of the individual (self-mode) H-G functions. High order modes propagate more slowly than low order modes so the field changes with axial position. A periodic re-imaging of the field occurs exactly as predicted by the ray propagation model.

A little mathematical notation can be used to summarise the situation. Firstly the propagation of each mode along the axis can be shown to be of the form

$$E_{lm}(x,y,z)=\Psi_{lm}(x,y)\exp(i\beta_{lm}z)$$

The field of each mode is E_{lm}. The basic H-G mode is Ψ_{lm} where l and m represent the order of the mode. A propagation factor for each mode is β_{lm} which represents the different speed of propagation as a function of l and m. There is clearly a periodic re-imaging over a distance $z=4L$ such that $4\beta_{lm}L=2\pi$. It can be shown that over a distance $z=L$ the input field is Fourier transformed. Over multiples of the axial distance L the input field is modified exactly in accordance with multiple applications of the Fourier transform. A possible definition of the FractFT is now apparent: the FractFT is the transform corresponding to propagation of a field along axial distances a fraction of the length L. Such a definition automatically satisfies the basic requirements suggested in Part 1 of this article, namely additivity, identity and commutativity. These three properties do not uniquely define the FractFT[18], however, propagation in GRIN media corresponds to the most interesting and useful definition. An equivalent property of the FractFT can be specified to ensure uniqueness. In the next section rotation of the Wigner distribution will be considered as such a property.

Interestingly the H-G polynomial representation includes an arbitrary scale factor in the Gaussian width. This scale factor re-emerges as a complication in the FractFT scaling property.

In summary, the FractFT can be defined analogously to the propagation of an (optical) electric field in a GRIN lens/fibre. The order of the FractFT here is directly proportional to the propagation distance. Propagation over successive increments is equivalent to propagation over the total of those increments. This behaviour is paralleled by the additive and commutative properties of the FractFT operator. Finally it is worth mentioning that Gauss-Laguerre polynomials are more convenient for studying propagation in systems with axial symmmetry, but most FractFT papers concentrate on H-G modes.

The Fractional Fourier Transform and the Wigner Distribution Function

The Wigner distribution function (WDF) is a concept which occurs naturally in the study of quantum mechanics. The Wigner distribution has also been found useful in the analysis of optical propagation problems [19, 20]. But, perhaps, the most important application of the WDF is in signal processing where the time and frequency analyses are required simultaneously. Conventionally a signal can be represented as function of time, and alternatively represented as a function of frequency (via the Fourier Transform). In certain situations, such as speech analysis, sonar, and radar, for example, the change in frequency spectrum over time is of interest. Strictly speaking, the frequency spectrum cannot be a function of time because Fourier analysis requires integration of the signal over all time. To get around this major difficulty a Short Time Fourier Transform (STFT) has been proposed with limited success[21]. The WDF presents another way of viewing time and frequency properties simultaneously with the added benefit of several mathematically tractable properties (which will not be considered here). The WDF of a one dimensional function is defined as follows:

$$W(x,u)= \int_{-\infty}^{\infty} g(x'-x/2)\, g^*(x'+x/2)\, \exp(-2\pi i u x')\, dx'$$

So x is a measure of the overlap of a function g with its complex conjugate g*. The parameter u is, as before, the spatial frequency. In signal analysis the corresponding parameters are time and frequency rather than position and spatial frequency. Time varying spectra can have very *distinctive* signatures when viewed in terms of their WDFs. The classic example given is a linear chirp – a tone with a linearly increasing frequency over a given time period. Figure 2 shows the WDF of a linear chirp. Some care and practice is needed for the correct interpretation of WDFs but in this very simple case the change in frequency with time is clear. Also shown in figure 2 are the WDFs of a single, pure tone $g(x)=\exp(-2\pi i u_0 x)$ and a single impulse $g(x)=\delta(x-x_0)$. It can be readily shown that the definition of the FractFT developed so far has a very simple interpretation in terms of the WDF. Simply stated, the WDF of a FractFT is identical to the WDF of the original function except for a rotation in the x–u plane. The amount of rotation is proportional to the fractional order. So, for order a = 1 the WDF is rotated $\pi/2$ radians; for order a = 4 the rotation is 2π (or 360°). Using the WDF an order a FractFT can be visualised as a (π a/2) radian rotation.

A potential use of the FractFT can now be seen. By suitably rotating the WDF of a chirp signal, the WDF can be made either a delta function in variable x or in variable *u*. In the former case a FractFT of suitable order converts a chirp signal into an impulse with a location related to the chirp parameters. Three immediate applications are obvious, i) to de-chirp signals, ii) to chirp signals, and iii) to compress or decode certain categories of signal. The GRIN media of the previous section provide a way of optically implementing such signal processing strategies. Optical processing can avoid the significant computational effort required to numerically evaluate either the FractFT or the FFT (Fast Fourier Transform), so very fast processing speeds may be attained.

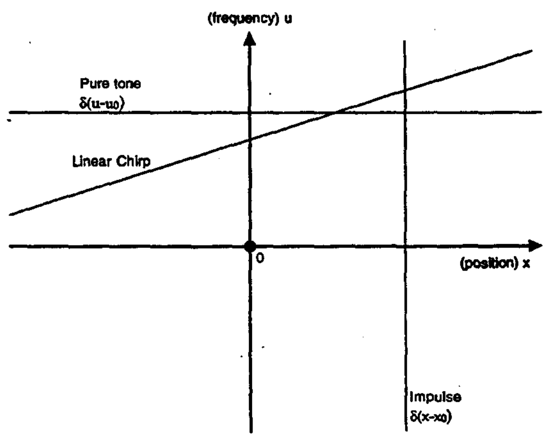

Figure 2 Wigner Distribution Function

The Fractional Fourier Transform and The Fresnel Transform

The Fresnel Transform is a little known transform[22] which can be defined using Fresnel diffraction in the same way that the Fourier transform can be defined using Fraunhoffer diffraction. Essentially, the Fresnel transform predicts the field in any plane parallel to the focal plane of the optical system shown in Figure 3. It has been shown that the Fresnel transform is equivalent to a suitable scaled version of theFractFT multiplied by a quadratic phase factor[23]. The equivalence is limited to a range of fractional orders typically 0 < a < 2 (a=1 can be a problem!). This is just another way of stating the obvious; that Fresnel diffraction does not exhibit periodic refocussing typical of GRIN media.

A single lens system can be used to realise some properties of the FractFT. In many cases the additional quadratic phase factor associated with the Fresnel transform is not important because it is the intensity (modulus squared of the field) which is detected. When the quadratic phase prevents direct evaluation of the FractFT, additional lenses (one at the input plane , one at the output plane) can be added to counterbalance this phase. Alternatively measurements may be limited to spherical surfaces which exactly counterbalance the phases again. An added difficulty in the FractFT interpretation of the Fresnel transform is that the fractional order is a monotonic but nonlinear function of axial distance.

A number of systems have been proposed for the optical implementation of the FractFT. Just two design criteria (apart from nominal focal strength) have to be addressed. These are i) the scaling factor, and ii) the quadratic phase factor. Both of these are related to the chosen fractional order. Readers interested in optical systems for FractFT processing should consider a number of references[14, 24-31].

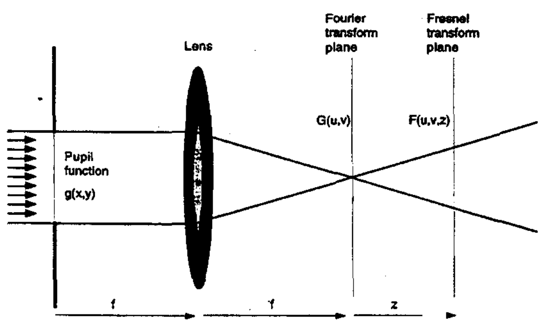

Figure 3 Fraunhofer and Fresnel Diffraction

Generalisations of the Fractional Fourier Transform

Just as the FractFT is an extension of the Fourier transform, the special affine Fourier transform[32, 33] or SAFT extends the FractFT. Here again the Wigner space viewpoint is most enlightening and calculus free (unless you want to prove it!). If the FractFT corresponds to a rotation in Wigner space then the SAFT corresponds to an arbitrary combination of shear, shift and rotation in Wigner space.

As mentioned earlier, free space propagation of field corresponds to a positional (or x) shear in Wigner space whereas the action of a lens corresponds to a frequency (or u) shear. A combination of three shears, such as x-u-x, can produce a pure rotation and hence a simple lens can produce a FractFT.

The SAFT may be useful as an alternative way to view and interpret field propagation in more general optical systems. Such an alternative may provide insight into seemingly intractable problems.

The Fractional Fourier Transform and Ray Propagation: The Fractional Legendre Transform

Propagation of a ray through an optical system can be represented - in a plane at least - by the ray position (r) and the ray angle (s). In figure 1 the ray height follows a sinusoid while the ray angle follows a cosinusoid as it propagates in the z direction. If the propagation of a ray bundle is plotted in a r-s diagram (figure 4) then propagation in a GRIN medium is seen to produce a rotation about the origin. The analogy with propagation and the WDF is strong.

The actions of a lens and free space propagation are shears in the s and r directions, respectively, further strengthening the analogy. The FractFT allows a function of position to be represented as a function of position and spatial frequency. In a similar way, a construction now called the fractional Legendre transform[34] allows a function of ray position to be represented as a function of a new parameter. This new parameter is a linear combination of both position and angle.

The useful application of such a transform occurs in the Hamiltonian optics of certain optical systems which exhibit singularities of both the point (ie position) characteristic and the angle characteristic. The fractional transform allows a new intermediate characteristic type some where between the point and the angle characteristic to be defined so that a singularity is avoided.

Applications of the Fractional Fourier Transform

The two main applications mentioned so far are:

A) Alternative interpretation of optical propagation in both GRIN media and conventional optical systems, and

B) Processing of signals with specific time-frequency and space-frequency signatures.

In many cases the FractFT is just used as a conceptual aid to help visualize certain aspects of propagation. It should be mentioned that the FractFT is based on a paraxial scalar approximation not unlike that of the Fresnel transform. So, in systems with high numerical aperture (such as microscopes) the approximation is invalid.

Recently some computational aspects of the FractFT have been investigated[10, 35]. It seems that the discrete FractFT can be used for the efficient calculation of the FFT of sparse datasets[11]. The well known trick of zero

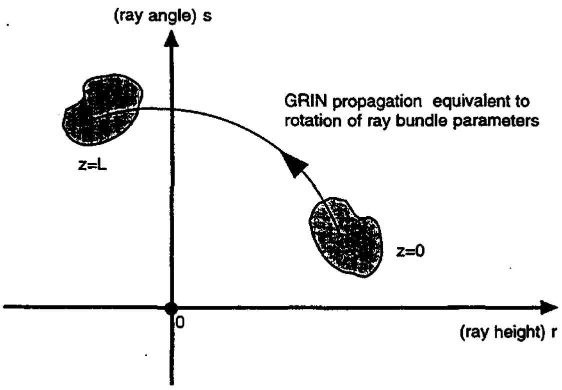

Figure 4 Ray height-angle (r-s) diagram

padding a dataset to get improved resolution in the FFT can be modified so that all redundant zero multiplications are avoided. The method also allows the calculation to be confined to a selected region of frequency space, further improving efficiency. Ironically this may mean that the calculation of Fresnel diffraction in optical systems can be speeded up considerably allowing improvements in calculation time and/or accuracy.

Finally...

Observant readers may have noticed that a mathematical definition of the FractFT has appeared nowhere in this article. The omission is deliberate.

Cynical readers may have noticed that the FractFT is a surefire method of generating periodic citations. I believe, as might less cynical readers, that the rapid dissemination of these ideas will have a beneficial effect.

Anagrammatical readers will surely have noticed the connection between FRACTIONAL FOURIER TRANSFORM and ASTRONOMIC NARRATOR FLUFFIER as well as REFRACTION AFFIRMS LUNAR ROOT.

Most other readers will have completely missed this article. If, by some chance, you have read this far please confirm my hopes/worst fears and email your comments to: K.Larkin@physics.usyd.edu.au

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I would like to thank Colin Sheppard for numerous stimulating discussions. John Sheridan provided some insight into the conceptual uses of the FractFT and also a number of useful references. Thanks are due also to Miguel Alonso for numerous obscure references which further undermine the accepted view of the FractFT's historical development. There is always a sneaking suspicion that the FractFT was originally conceived (along with the FFT) by C.F. Gauss and hidden in some obscure neo-classical latin manuscript.@

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