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Similarity theorems for fractional Fourier transforms and fractional Hankel transforms

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Abstract

The significance of the similarity theorem for the fractional Fourier transform is discussed, and the properties of self-similar functions considered. The concept of the fractional Hankel transform is developed for use in the analysis of diffraction and imaging in symmetrical optical systems. The particular case of Fresnel diffraction from a circular aperture is discussed and the effects of the similarity theorem are described. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The concept of the fractional Fourier transform has been rediscovered several times [1–9]. Recently, it is becoming of importance in various applications of optics including graded-index optics [8], optical information processing [9–11] and free-space propagation [12].

A detailed account of the properties of the fractional Fourier transform was given by Namias [3]. He gave an explicit expression for the transform, some examples of transforms of simple functions, and details of various theorems analogous to those of the Fourier transform. Many of the early papers were mainly concerned with applications in quantum mechanics and hence used an operator formalism. Namias [3] mentions that the similarity rule can be established, but that unfortunately it is not of simple form.

Gori [12] has shown the connection between the fractional Fourier transform and the Fresnel transform. He also considers [13] why the Fresnel transform is not widely used. Perhaps in optics the answer is that the Fresnel transform representation does not extend easily to the important cases, diffraction in axially symmetric systems, or imaging of axially symmetric objects.

If we consider Fresnel diffraction, the diffracted field can be expressed in terms of optical coordinates, representing transverse and axial distances normalized by the wavelength and some characteristic dimensions. The axial distances depend on the scale of the transverse distances. This property is related to the non-uniqueness of fractional Fourier order, and suggests that the fractional Fourier transform may often be a more convenient approach to investigating diffraction problems.

There are very few plots given of fractional Fourier transforms in the literature. A complication is that the order of a fractional Fourier transform depends on the scale of the original function. Ozaktas and Mendlovic [9] have considered the fractional Fourier transform of a δ-function. However, their results do not agree with those of Namias [3], who shows that the magnitude for any non-integer fractional order is independent of the argument. We believe that the results given by Mendlovic really apply for a rectangular pulse of finite width. The small size of the pulse results in the fractional Fourier transform, for intermediate values of order, corresponding to the near-field of the pulse.

Although many papers have mentioned the application of the fractional Fourier transform to optical propagation,

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and a number of papers [14,15] have rigorously derived the fractional Hankel transform, little appears to have been published on the fractional Fourier transform applied to axially symmetric optical systems. Here we discuss the similarity theorem for this case, which leads to a simple investigation of the fractional Fourier transform of a circular aperture. A similar extension to the case of a spherically symmetry function is also straightforward.

2. The similarity theorem for fractional Fourier transforms

The fractional Fourier transform can be written in operator notation [3]

$$F_\alpha f(x) = \frac{\exp[\mathrm{i}(\pi/4-\alpha/2)]}{\sqrt{2\pi\sin\alpha}}\exp\left(-\frac{\mathrm{i}x^2}{2}\cot\alpha\right)\times\int_{-\infty}^{+\infty}\exp\left(-\frac{\mathrm{i}x'^2}{2}\cot\alpha\right)\times\exp\left(\frac{\mathrm{i}xx'}{\sin\alpha}\right)f(x')\,\mathrm{d}x', \tag{1}$$

in which the order of the transformation is $n=2\alpha/\pi$.

This form has the important properties that for $\alpha=\pi/2$ it reduces to the ordinary Fourier transform, that if applied twice in succession the final order is the sum of the individual orders, and that it satisfies the generalization of Parseval's theorem.

We require to find the fractional Fourier transform of the function

$$f(x') = g(x'/c) = g(x''), \tag{2}$$

which can be written

$$F_\alpha f(x) = \frac{|c|\exp[\mathrm{i}(\pi/4-\alpha/2)]}{\sqrt{2\pi}\sin\alpha}\exp\left(-\frac{\mathrm{i}x^2}{2}\cot\alpha\right)\times\int_{-\infty}^{+\infty}\exp\left(-\frac{\mathrm{i}x''^2}{2}\cot\beta\right)\times\exp\left(\frac{\mathrm{i}x''y}{\sin\beta}\right)g(x'')\,\mathrm{d}x'', \tag{3}$$

where

$$c^2\cot\alpha=\cot\beta,\quad \frac{xc}{\sin\alpha}=\frac{y}{\sin\beta}. \tag{4}$$

By considering the fractional Fourier transform of order β, we find

$$F_\beta f(cy) = \sqrt{\frac{\sin\alpha}{\sin\beta}}\frac{1}{|c|}\exp\left[\frac{\mathrm{i}}{2}(\alpha-\beta)\right]\times\exp\left[\frac{\mathrm{i}}{2}(x^2\cot\alpha-y^2\cot\beta)\right]F_\alpha f(x). \tag{5}$$

Eliminating x and either c or α, we obtain two different forms for the similarity theorem:

$$F_\beta f(cy) = \sqrt{\frac{1-\mathrm{i}\tan\beta}{1-\mathrm{i}\tan\alpha}}\exp\left[\frac{-\mathrm{i}y^2\cot\beta}{2}\left(1-\frac{\cos^2\alpha}{\cos^2\beta}\right)\right]\times F_\alpha f\left(y\sqrt{\frac{\sin2\alpha}{\sin2\beta}}\right), \tag{6}$$

$$= \sqrt{\frac{1-\mathrm{i}\tan\beta}{1-\mathrm{i}c^2\tan\beta}}\exp\left[\frac{-\mathrm{i}y^2}{2}\left(\frac{c^4-1}{c^4\tan\beta+\cot\beta}\right)\right]\times F_\alpha f\left(\frac{cy}{\sqrt{c^4\sin^2\beta+\cos^2\beta}}\right). \tag{7}$$

We see that the fractional Fourier transform of a scaled function experiences a different scale to that of the original function, and is multiplied by a parabolic phase term, but more importantly also experiences a different order. These effects are consistent with our view in diffraction that a change of scale of an aperture alters the Fresnel number. A similar relation to Eq. (6) was presented by Alieva et al. [16] but was not discussed further.

We point out that some functions exhibit a property of self-similarity (or scale invariance), in the sense that when scaled they change only by, perhaps, a constant factor. Examples of such functions include the signum function, the Heaviside step function, the Dirac δ-function, a constant, or x raised to some power. It is immediately apparent that the fractional Fourier transform of a delta function,

$$F_\alpha \delta(x) = \frac{\exp[\mathrm{i}(\pi/4-\alpha/2)]}{\sqrt{2\pi\sin\alpha}}\exp\left(-\frac{\mathrm{i}x^2}{2}\cot\alpha\right), \tag{8}$$

satisfies the self-similarity condition in Eq. (5).

An important case in optics, which does not seem to have been discussed elsewhere, is the Heaviside step function, defined as

$$F(x) = \begin{cases}1, & x>0,\\ 0, & x<0.\end{cases} \tag{9}$$

Substituting into Eq. (1) we find that

$$F_\alpha f(x) = \sqrt{1-\mathrm{i}\tan\alpha}\,\exp\left(\frac{\mathrm{i}x^2}{2}\tan\alpha\right)\times\left[\frac{1}{2}-\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}}F\left(\sqrt{\frac{2}{\pi\sin2\alpha}}\,x\right)\right], \tag{10}$$

where

$$F(u) = C(u) - \mathrm{i}S(u) \tag{11}$$

is a complex Fresnel integral. We can readily see from Eqs. (5) and (6) that

$$\frac{\cos\alpha}{\cos\beta} = \frac{1}{c^2}\frac{\sin\alpha}{\sin\beta} \tag{12}$$

and

$$\frac{x^2}{\sin2\alpha} = \frac{y^2}{\sin2\beta}, \tag{13}$$

so that Eq. (13) again satisfies the similarity requirement.

The fact that for $\alpha \to 0$, the fractional Fourier transform reduces to the original function is also apparent. As α increases, the scale of the Fresnel integral also increases and then decreases, but remains given by a Fresnel integral for $\alpha < \pi/2$. This is an interesting parallel to the behaviour in diffraction. The Fresnel diffraction pattern of an edge can be expressed in terms of Fresnel integrals for any *finite* distance from the edge. Whereas the Fourier transform of a step function is not given by a Fresnel integral, correspondingly the associated Fraunhofer pattern is not observable in practice because of the infinite dimension of the aperture.

In general we can say that for any self-similar function the fractional Fourier transform for any order can be obtained by scaling the transforms and multiplying by a quadratic phase factor and a function of the order only.

Lohmann [10] has proposed two different optical systems for performing a fractional Fourier transformation. The first of these uses a single converging lens with input and output planes equidistant on either side of the lens. The input plane is illuminated by a plane wave. The second consists of two equal converging lenses placed immediately after the input plane and immediately before the output plane. It is interesting to note that these two arrangements are analogous to the two different ways of calculating Fresnel diffraction: the first is similar to a Fourier transformation followed by a parabolic phase change and then another Fourier transformation, which is equivalent to the angular spectrum method. The second is similar to a phase change followed by a Fourier transformation and another phase change, which is equivalent to the direct Fresnel approach to performing a Fresnel transformation.

We consider the first arrangement, with a lens of focal length f, and input and output planes distant z from the lens. Then the output amplitude is given by Fourier optics as

$$u(x) = -\frac{e^{2ikz}}{\lambda^2 z^2}\int_{-\infty}^{+\infty} u'(x')\exp\left[\frac{ik(f-z)}{2z(2f-z)}(x^2+x'^2)\right] \times\exp\left[\frac{-ikfxx'}{z(2f-z)}\right]dx'. \tag{14}$$

Comparing the exponents with Eq. (1) we obtain two equations if these are to be equivalent.

Introducing a^2k/f, an arbitrary dimensionless scaling factor which determines the relative dimensions in the two equations,

$$\cos\alpha = 1-\frac{z}{f},\quad a\sqrt{\frac{k}{f}} = \left[\frac{z}{f}\left(2-\frac{z}{f}\right)\right]^{1/4}, \tag{15}$$

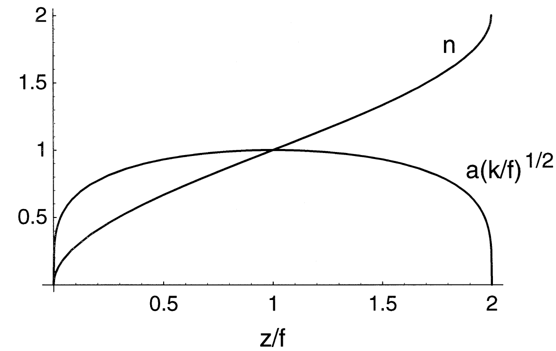

Fig. 1. The order of fractional Fourier transformation and the scaling factor for an optical implementation.

which are plotted in Fig. 1. The order of the transformation can be altered in the range from zero to two by changing z/f, but of course the relationship is nonlinear. Further, it is seen that the relative scaling of the fractional Fourier transform alters as z/f is changed. This means that if this simple optical system is to be used to provide transformations of different orders, masks of different scales must be employed. This is obviously a serious limitation of the practical implementation of this approach.

3. The fractional Hankel transform

Extending the fractional Fourier transform to two dimensions and introducing cylindrical coordinates allows us to introduce the fractional Hankel transform of a particular angular order. Here we shall concentrate on the important case of axial symmetry, corresponding to diffraction by an axially symmetric structure, or imaging of axially symmetric objects, when we can perform the angular integral to give for the zero angular order fractional Hankel transform

$$H_\alpha f(r) = \frac{i\exp(-i\alpha)}{2\sin\alpha}\exp\left(-\frac{ir^2}{2}\cot\alpha\right) \times\int_0^\infty 2\exp\left(-\frac{ir'^2}{2}\cot\alpha\right) \times J_0\left(\frac{rr'}{\sin\alpha}\right) f(r')\,r'\,dr', \tag{16}$$

where J_n is a Bessel function of the first kind of order n. Here we prefer to use a slightly different definition of the parameter α from that of Namias and Kerr [14,15]. Our development may be alternatively viewed as an extension of the Hermite–Gaussian representation in one and two dimensions to the Laguerre–Gaussian representation of two dimensional systems with axial symmetry (see Ref. [17] for example).

An important example in optics is the circular aperture

$$f(r) = \begin{cases} 1, & r \le 1 \\ 0, & r > 1 \end{cases} \tag{17}$$

giving

$$H_\alpha f(r) = \frac{\mathrm{i}\exp(-\mathrm{i}\alpha)}{2\sin\alpha} \exp\left(-\frac{\mathrm{i}r^2}{2}\cot\alpha\right) \times \int_0^1 2\exp\left(-\frac{\mathrm{i}r'^2}{2}\cot\alpha\right) \times J_0\left(\frac{rr'}{\sin\alpha}\right) f(r')\, r'\,\mathrm{d}r' \tag{18}$$

$$= \frac{\mathrm{i}\exp(-\mathrm{i}\alpha)}{2\sin\alpha} \exp\left(-\frac{\mathrm{i}r^2}{2}\cot\alpha\right) I\left(\cot\alpha, \frac{r}{\sin\alpha}\right), \tag{19}$$

where $I(u,v)$ is defined by

$$I(u,v) = 2\int_0^1 \exp\left(-\frac{\mathrm{i}u\rho^2}{2}\right) J_0(v\rho)\,\rho\,\mathrm{d}\rho. \tag{20}$$

We now recognize the relationships, for this particular scaling, between the fractional Hankel transform coordinates α, r and the optical coordinates u,v introduced by Born and Wolf [18],

$$u = \cot\alpha, \quad v = r/\sin\alpha. \tag{21}$$

The fractional Hankel transform can thus be written in terms of the optical coordinates

$$H_\alpha f(r) = \frac{1+\mathrm{i}u}{2} \exp\left[-\frac{\mathrm{i}uv^2}{2(1+u^2)}\right] I(u,v). \tag{22}$$

Of course aberrations and apodization can be very simply introduced into the expression for $I(u,v)$. For the aberration-free case the integral can be evaluated in Lommel functions

$$I(u,v) = \frac{2}{u}\exp(-\mathrm{i}u/2)\left(U_1(u,v) + \mathrm{i}U_2(u,v)\right) \tag{23}$$

$$= -\frac{2\mathrm{i}}{u}\left[\exp\left(\frac{\mathrm{i}v^2}{2u}\right) + \exp(-\mathrm{i}u/2)\left(V_0(u,v) - \mathrm{i}V_1(u,v)\right)\right], \tag{24}$$

these expressions being convenient for points outside and inside the geometrical shadow edge respectively, which is defined by

$$r = \cos\alpha. \tag{25}$$

We also have the special cases

$$I(0,v) = \frac{2J_1(v)}{v}, \tag{26}$$

$$I(u,0) = \exp\left(-\frac{\mathrm{i}u}{4}\right)\frac{\sin(u/4)}{u/4}, \tag{27}$$

and

$$I(u,u) = \frac{\mathrm{i}}{u}\exp\left(\frac{\mathrm{i}u}{2}\right)\left(1 - \exp(-\mathrm{i}u)J_0(u)\right), \tag{28}$$

so that

$$H_{\pi/2} f(r) = \frac{J_1(r)}{r}, \tag{29}$$

$$H_\alpha f(r)\big|_{r=0} = 2(1 - \mathrm{i}\tan\alpha)\exp\left(-\frac{\mathrm{i}\cot\alpha}{4}\right)\sin\left(\frac{\cot\alpha}{4}\right), \tag{30}$$

and

$$H_\alpha f(r)\big|_{r=\cos\alpha} = \frac{1}{2}(1 - \mathrm{i}\tan\alpha)\exp\left(-\frac{\mathrm{i}\sin 2\alpha}{4}\right) \times\left[1 - \exp(-\mathrm{i}\cot\alpha)J_0(\cot\alpha)\right]. \tag{31}$$

The modulus squared of the latter two variations are illustrated in Figs. 2 and 3. Fig. 2(b) shows that the

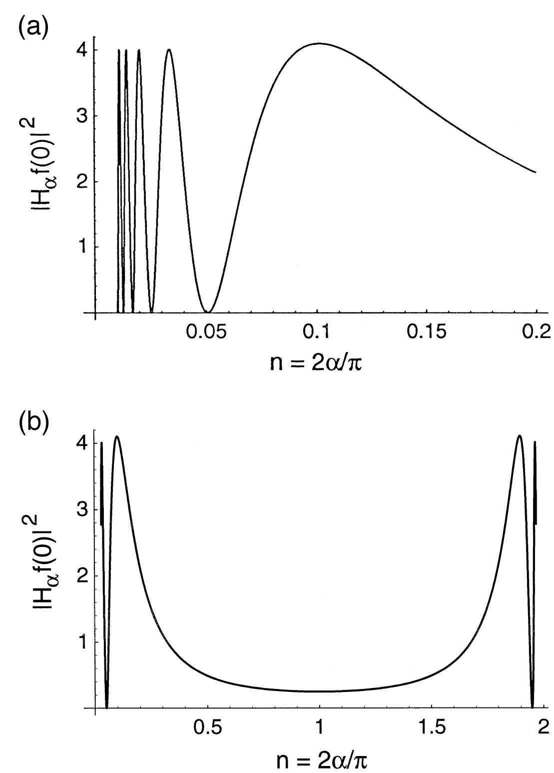

Fig. 2. The axial variation of the modulus squared of the fractional Hankel transform of order n of a circular aperture of radius unity: (a) $0 \le n \le 0.2$, (b) $0 \le n \le 2$.

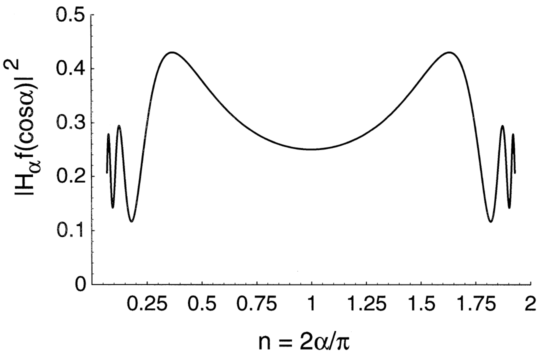

Fig. 3. The variation along the shadow edge of the modulus squared of the fractional Hankel transform of a circular aperture of radius unity.

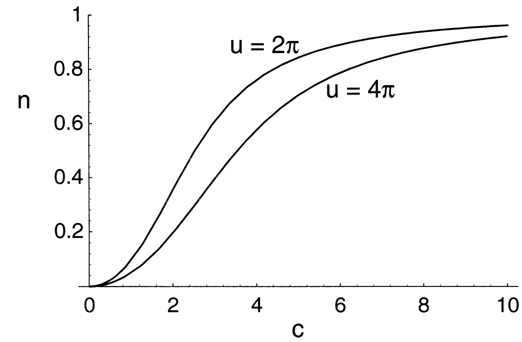

Fig. 5. The order of fractional Hankel transform for the first maximum $(u \approx 2\pi)$ and the first zero $(u = 4\pi)$ as a function of the radius c of the circular aperture.

modulus squared of the variation with transformation order for $r = 0$ oscillates between values of zero and 4, and that for $\alpha = \pi/2$ it has a value of $1/4$. The function oscillates wildly on approaching $\alpha = 0$ or π. It is interesting to note that because of our particular choice of scaling factor and the very nonlinear behaviour of cot α, most of the range of orders corresponds to situations close to the Hankel transform of order unity. Values of α close to zero and π correspond to situations where the Fresnel approximation tends to break down. In Fig. 2(a) the peak closest to the centre, i.e. $n = 1$, corresponds to the case when $u \sim \pm 2\pi$, which is the primary focus of a pinhole lens. The zeros closest to the centre correspond to $u = \pm 4\pi$, which are the first zeros from focus along the axis of a convergent beam. These cases correspond to orders of transformation of ~ 0.1 and ~ 0.05 respectively. Fig. 4 shows contours of constant intensity of the fractional Hankel transform of a circular disc.

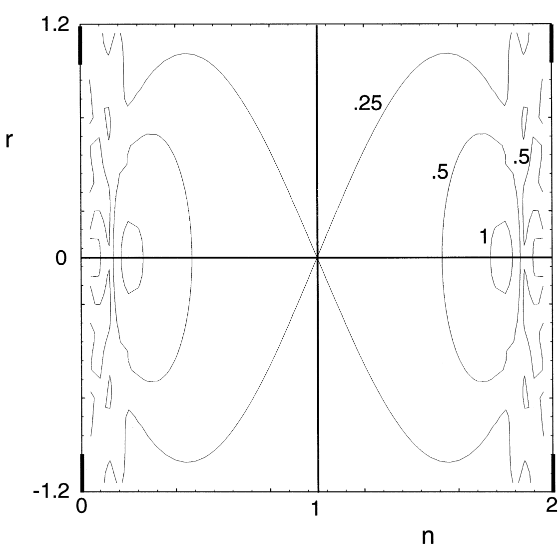

Fig. 4. Contours of constant intensity in the fractional Hankel transform of a circular disc of radius unity.

The optical coordinates are related to true distances according to the geometry of the arrangement. Besides the case of a focused beam considered by Born and Wolf, we also have the case of diffraction of a collimated beam by a circular aperture of radius a, for which

$$u = \frac{ka^2}{R}, \quad v = \frac{ka^2\rho}{R}, \qquad (32)$$

where $k = 2\pi/\lambda$, and ρ and R are cylindrical and spherical polar radii of the observation point [19]. This latter reference also gives details of other transformations which give improved accuracy of the Fresnel approximation for regions close to the aperture, for high numerical aperture, for finite values of Fresnel number, and for off-axis illumination.

A similarity theorem for fractional Hankel transforms can be developed exactly as for Fourier transforms. Here we shall just comment on the special case of a circular aperture of radius c. The first order Hankel transform is then, of course, shrunk by a factor, whilst the behaviour with order is distorted according to Eq. (5). In Fig. 5 we illustrate the effect of altering the value of c on the fractional order, n, of the fractional Hankel transform.

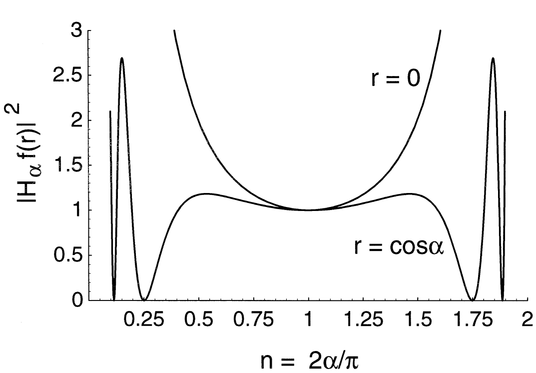

Fig. 6. The modulus squared of the fractional Hankel transform of a narrow ring aperture along the axis $(r = 0)$ and along the shadow edge $(r = \cos\alpha)$.

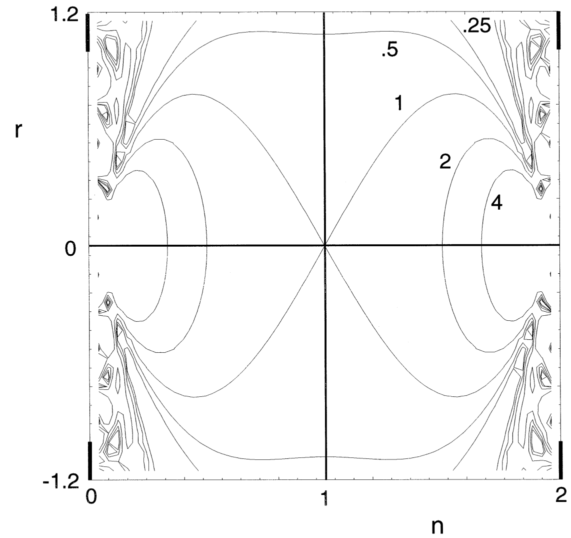

Fig. 7. Contours of constant intensity in the fractional Hankel transform of a circular ring of radius unity.

The fractional Hankel transform for an annular aperture can also be developed. Here we just mention the special case of a narrow ring aperture

$$f(r)=\frac{\delta(r-1)}{r}. \qquad (33)$$

The fractional Hankel transform is then

$$H_\alpha f(r)=(1+\mathrm{i}\cot\alpha)\exp\left(-\frac{\mathrm{i}}{2}(r^2+1)\cot\alpha\right)\times J_0\left(\frac{r}{\sin\alpha}\right). \qquad (34)$$

This represents the now misleadingly named ‘diffraction-free’ beam. The scale of the Bessel function thus shrinks on approaching $n=0$ or 2. The behaviour of the squared modulus of the transform along the axis and the shadow edge is shown in Fig. 6. There are no oscillations along the axis in this case. A contour plot of the fractional Hankel transform of a circular ring is shown in Fig. 7.

4. Discussion

The significance of the similarity theorem for fractional Fourier transforms has been investigated. In particular it is observed that the order of the transformation depends on the assumption of a particular scale for the function. The properties of self-similar functions have been discussed. The fractional Hankel transform has been presented in the context of Fresnel diffraction in circular symmetric optical systems, but also applies to imaging of axially symmetric objects. The common case of a uniform circular aperture has been developed in detail to show the correspondence between the fractional order and the optical coordinates of Born and Wolf [18]. In this case, the similarity theorem gives the fractional order as a function of both scale and axial optical coordinate.

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