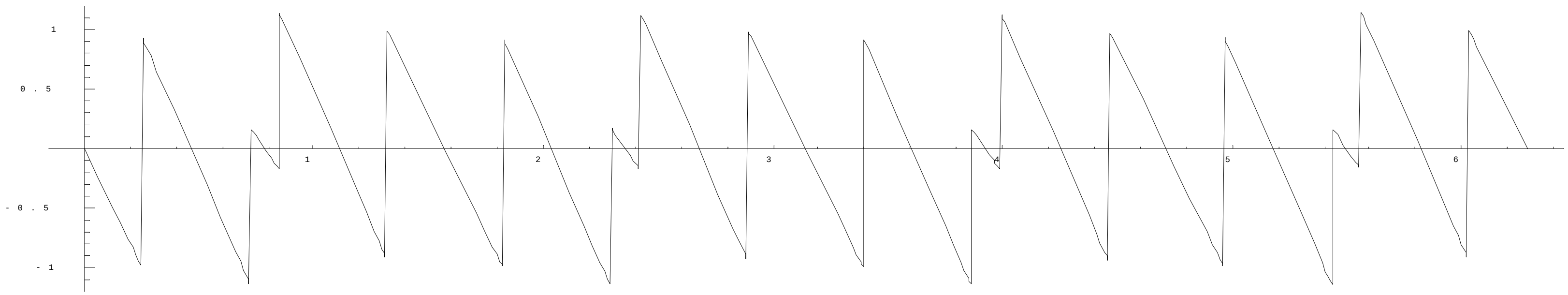
Finite, Tractable Formulae for



Correlated Quantisation Errors in Phase Measuring Interferometry

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Background

- Chris Brophy 1990
  - errors due to quantisation are not random
  - 90° step algorithm's are highly correlated

- Bing Zhao 1997
  - Bessel function expansion (slow convergence)
  - complicated!
  - More terms than quantization levels (4x)

Significance

- Phase-shifting algorithms (+interferometers) now so good that errors down near quantization levels in some cases.
- Error correcting algorithms and self-calibrating schemes can remove step errors
- Gravity wave optics and interferometry?

Errors in Phase-Shifting Algorithm

$$S = \sum_{m=1}^{M} s_m I_m$$

PHASE

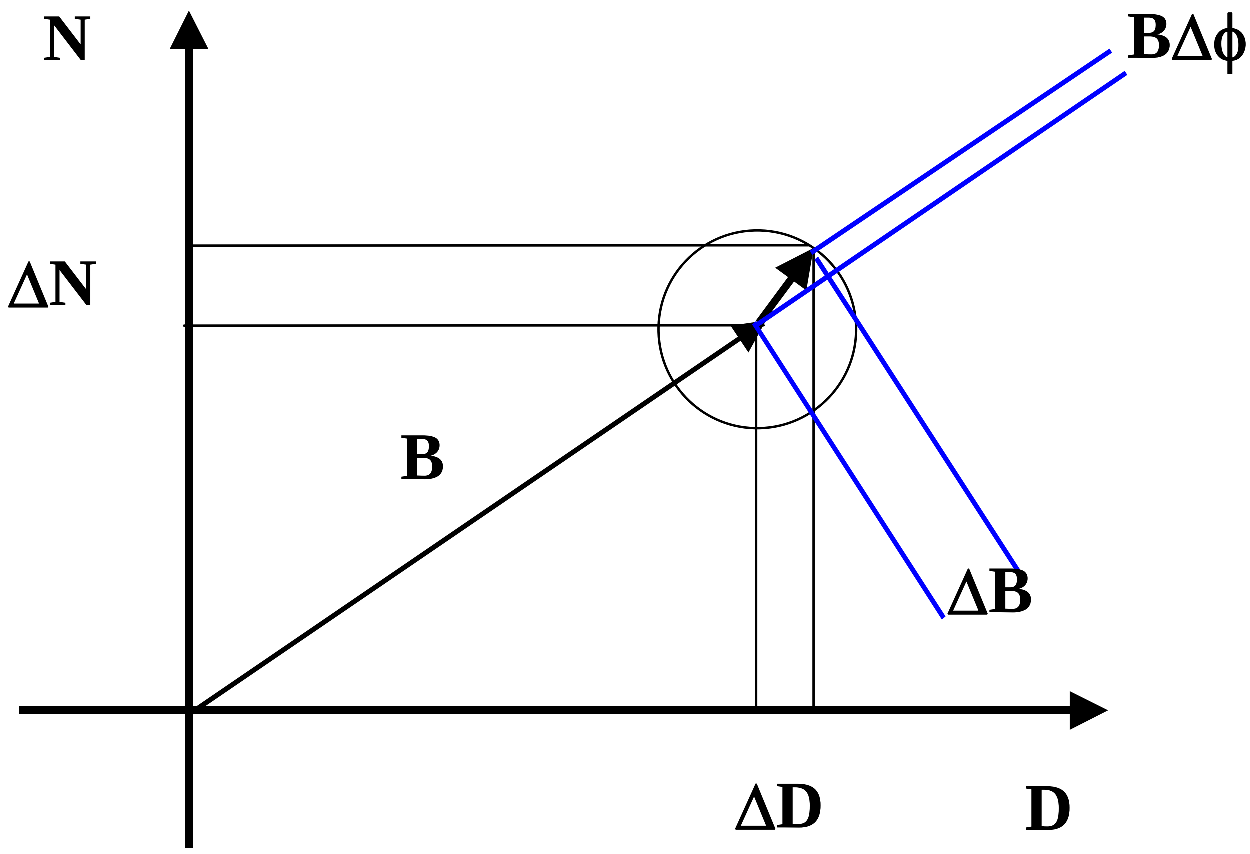
$$\tan\phi = \frac{S}{C}$$

$$C = \sum_{m=1}^{M} c_m I_m$$

MODULATION

$$B^2 = S^2 + C^2$$

Modulation and Phase Errors



Errors in Phase-Shifting

$$N = B\sin\phi$$

$$D = B\cos\phi$$

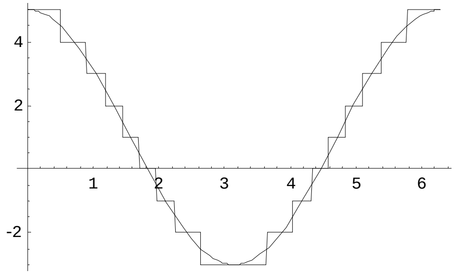
$$N^2 + D^2 = B^2$$

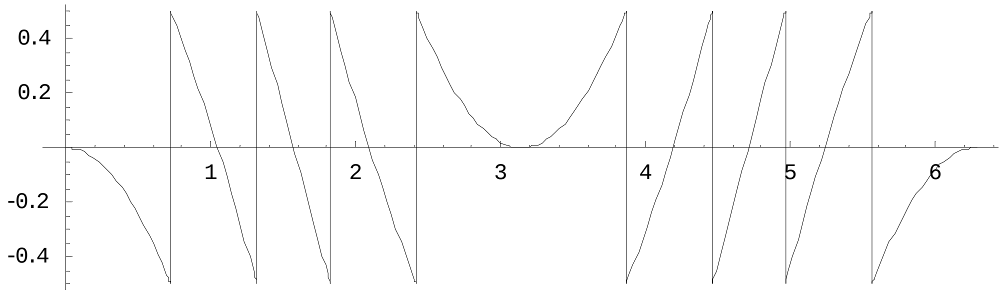
$$\Delta N = \Delta B.\sin\phi + B.\cos\phi.\Delta\phi$$

$$\Delta D = \Delta B.\cos\phi - B.\sin\phi.\Delta\phi$$

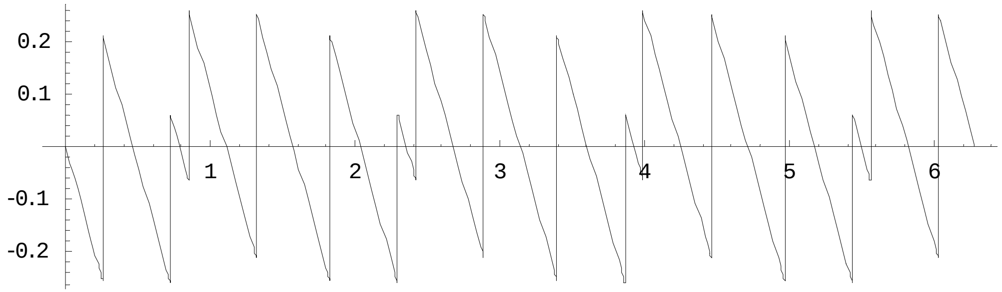
PHASE MODULATION

Quantization Errors

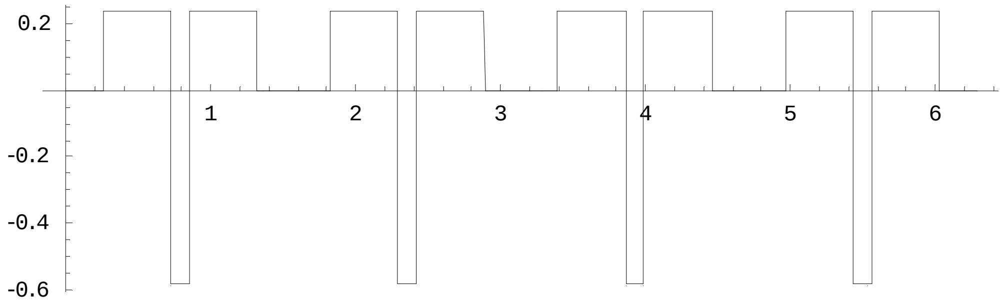




SIGNAL

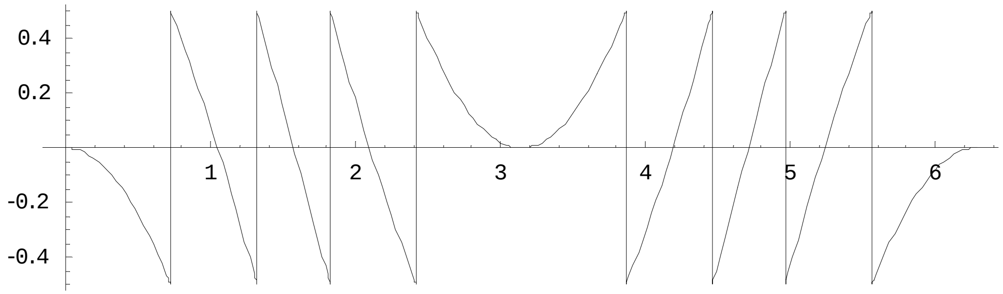


PHASE



MODULATION

Bessel Series Approach



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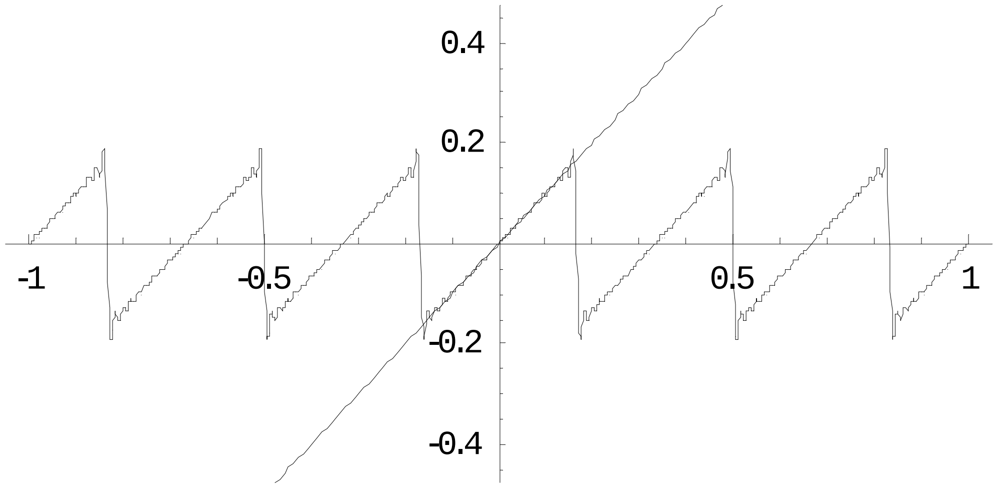


20 term series expansion of phase error

Convergence:

problem with numerous discontinuities

Exact Description

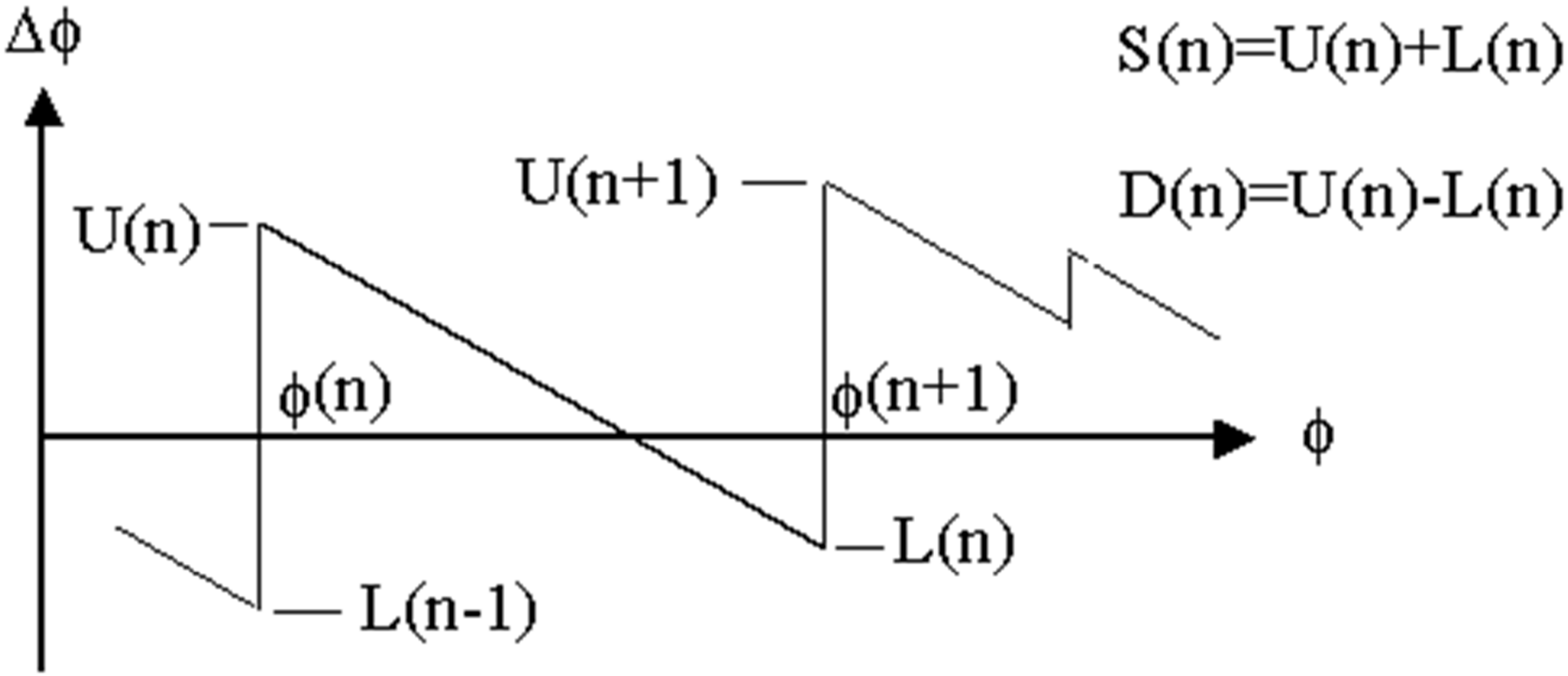


Figure 2. Phase error parameters

Can also specify approximately in terms of zero crossings (exact zero mean function).

Exact Calculation

$$0 = \sum_{n=1}^{J} S(n)$$

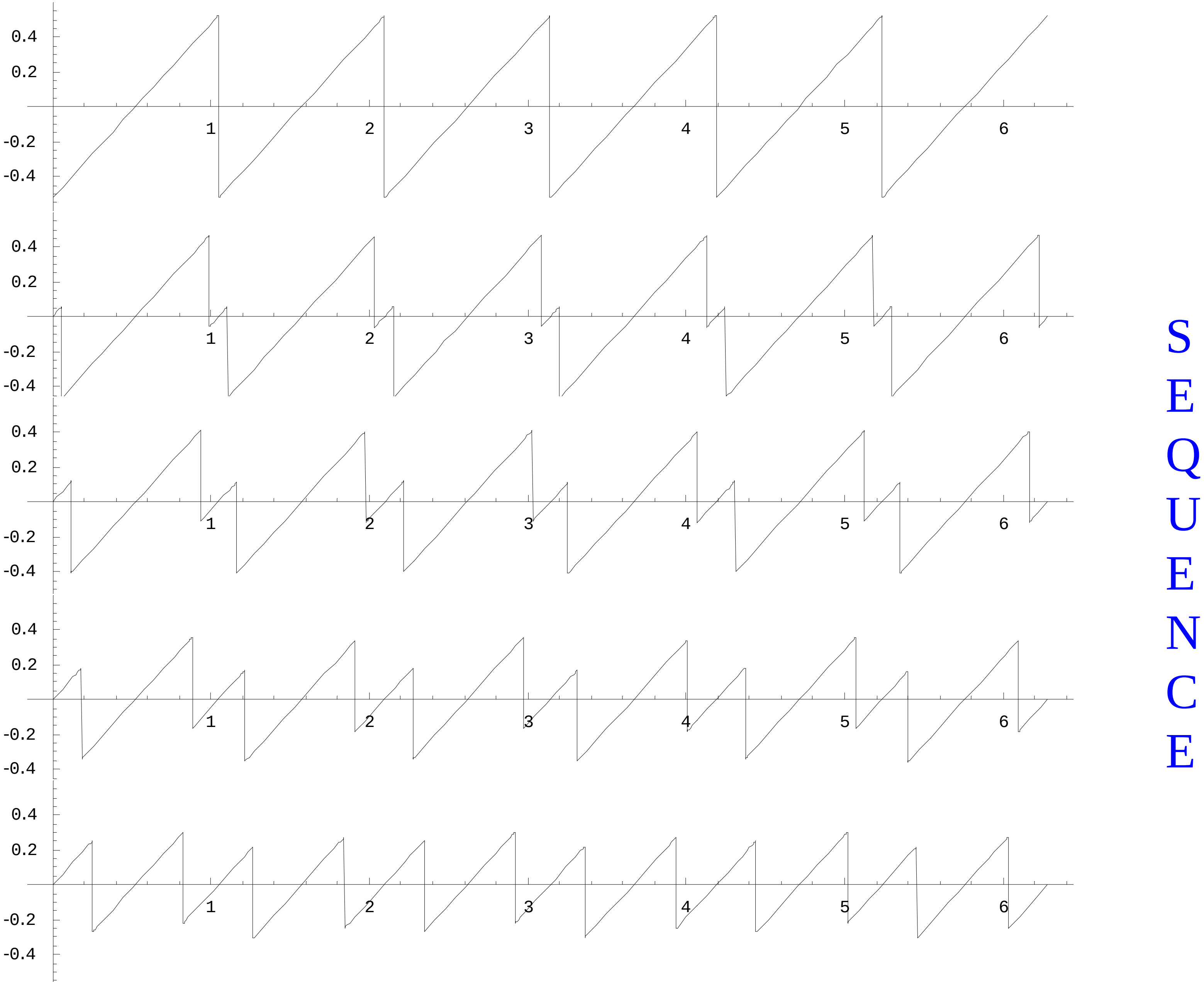
Sum of phases

$$2k\pi = \sum_{n=1}^{J} D(n)$$

Sum of phase differences

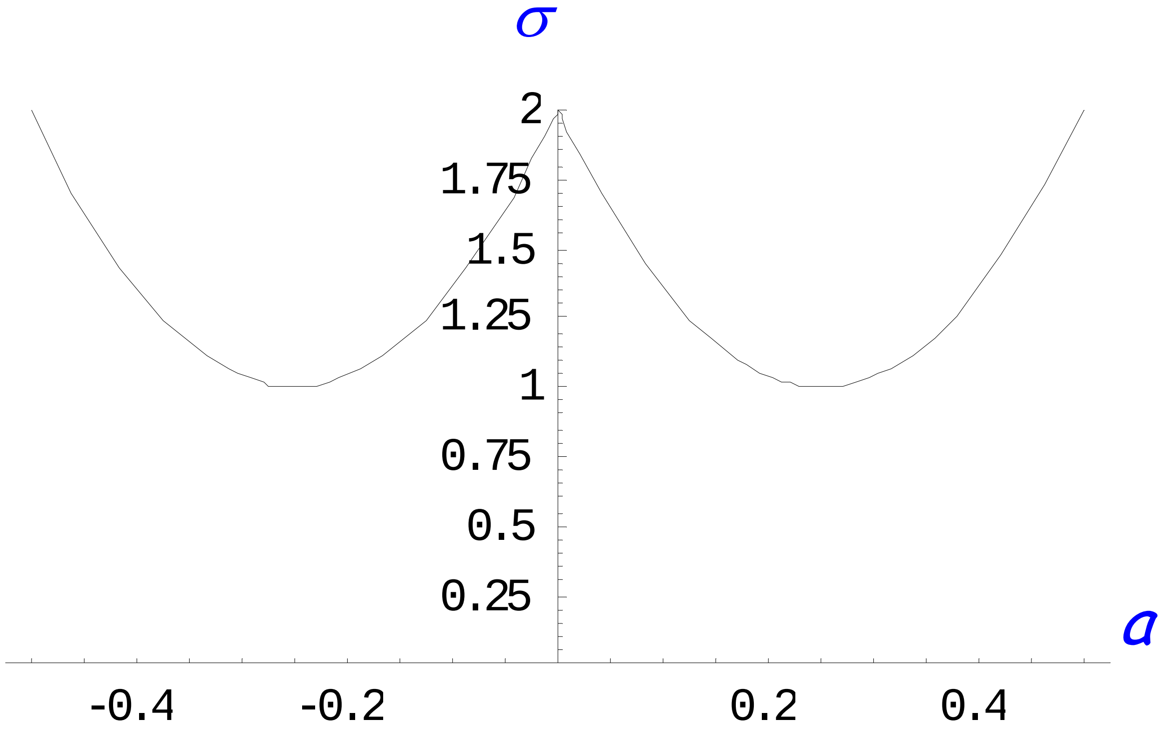
$$\sum_{n=1}^{J} [\Delta\phi(n)]^2 = \frac{1}{12} \sum_{n=1}^{J} D(n)\{3S^2(n) + D^2(n)\}$$

Variance of phase error

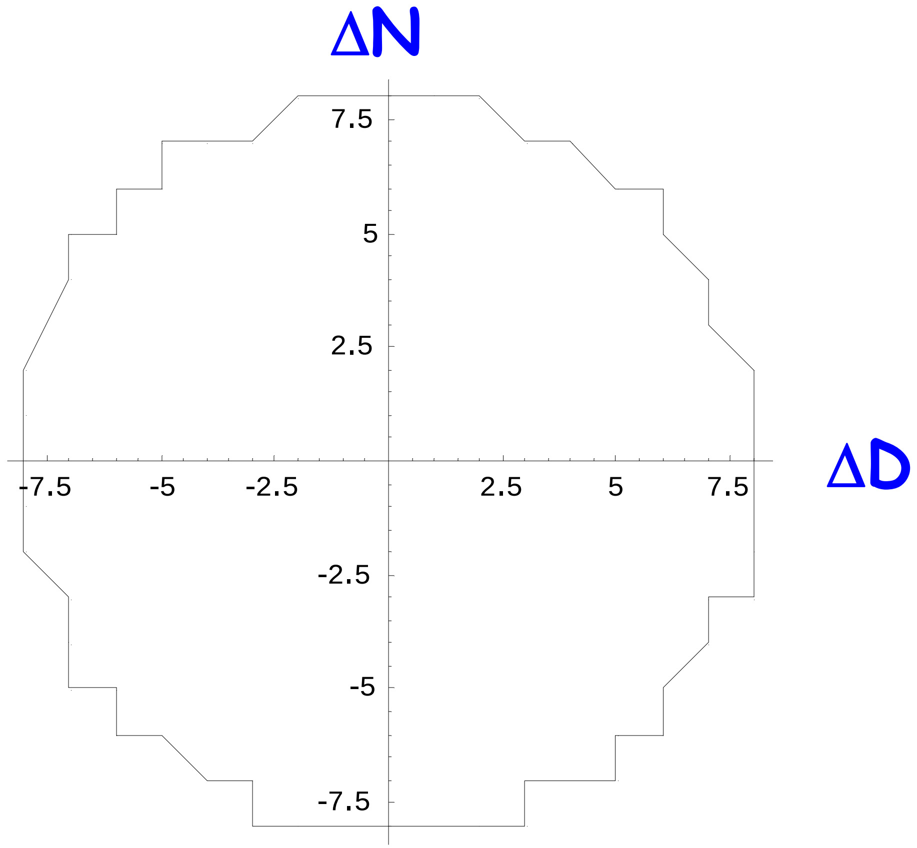


Correlation change with offset for 90° algorithms

Variation with offset *a*



Alternative Representations



Finite, Tractable Formulae for Correlated Quantisation Errors in Phase Measuring Interferometry

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Quantisation errors in phase-shifting interferometry are often ignored because they are well below the level of other noise sources. As the precision requirements increase, in areas such as quantum phase measurement and gravity wave interferometry, the quantisation errors may become more important.

In 1990 Brophy [1] investigated the effects of intensity quantisation on the output of phase-shifting algorithms (PSAs). In that work only a few specific algorithms are discussed and it is assumed that explicit evaluation of the phase variance is impractical if the number of quantisation levels is large. Subsequently a number of works by Zhao [2] concentrated on Fourier/Bessel series approaches in an effort to generalise the results. Unfortunately the analytic results are either slowly convergent infinite series, or multiple finite series with obscure terms which ultimately give little insight into the final, highly structured results.

We show that the variance can be calculated exactly in a small number of steps using a simple geometric interpretation. The series we derive contains (at most) NP terms, for an N sample PSA with P quantisation levels. This compares well to the series of Zhao with 4N^2P^2 terms.

Perhaps the most important observation to make is that the phase quantisation error has a particularly simple form. This fact seems to have been forgotten in recent work. Figure 1 shows a comparison of the intensity quantisation error versus the resultant phase quantisation error. Although the intensity error has a rather complicated sinusoid-like shape, the phase error is a straight sawtooth, albeit with variable spacing. Indeed it is the numerous oddly spaced discontinuities that make the Fourier series representation so slowly convergent. However, the constant slope sawtooth has many simple properties when it comes to evaluating means, means squares and other statistical parameters, without resorting to Fourier methods. The sawtooth profiles can be calculated as a weighted sum of intensity errors using well-known error propagation formulae [3].

To evaluate the statistical properties of the sawtooth it is generally necessary to locate the discontinuities resulting from each intensity component in the PSA. If the intensity has been quantised to P quantisation levels (P is typically ~256 in video analog-to-digital converters) and there are N frames or steps in the algorithm, then there will be NP discontinuities. For certain highly symmetrical algorithms, such as the N=4 PSA, some of these discontinuities are degenerate or coincident. The calculation of total phase error variance (for example) proceeds by a simple calculation of the parameter between every successive pair of discontinuities. The individual terms are then summed.

At first sight this approach does not seem to add any insight into the origin of the highly patterned correlations known to occur for phase error variance (or standard deviation). However, it is possible to obtain simple (ie non-series) expressions for the phase variance as a function of intensity offset if N is divisible by 4, for example.

If we define two values for each tooth of the sawtooth, the sum value S(n) and the difference value D(n) then the statistics become simple expressions:

Total of sum values = SUM{S(n)}=0

Total of diff. values = SUM{D(n)}=2*pi
Total of squares = SUM{D(n)*[3*S(n)^2+D(n)^2]/12}

The first two expressions are simple constraints upon the mean phase error and the mean phase error gradient. The last expression corresponds to the total phase error variance. The variance will be maximised where a few large values of D occur amongst many small (or zero) values of D, subject to the second constraint. This is actually equivalent to a sawtooth with a few widely spaced teeth with many smaller or zero width (ie coincident) teeth. Typically the 3*S(n)^2 term is small and positive. D(n) is always positive too, by definition. If we ignore the S(n) term for now, then the variance expression is a simple sum of cubed tooth widths. If we double the number of teeth then the average width halves and the variance drops by a factor of 4. This is exactly what happens in the classic case of shifting the intensity offset by 0.25 of a quantisation level Q, for the N divisible by 4 algorithms. For other algorithms the distribution of tooth widths is more complex and a simple halving does not occur. Figure 1 shows the intensity quantisation error with zero offset, followed by a plot with 0.25Q offset. The corresponding phase errors are then shown in sequence, and it is clear how the offset reduces the correlation.

A number of strategies for quantisation error reduction are suggested by our analysis. Three methods of immediate interest are:

i) intensity offset dithering (typically add or subtract 0.25 of quantisation level Q);
ii) intensity dithering (change overall intensity to give 0.25 Q over full range);
iii) phase-shift dithering (change phase steps slightly).

In all the above cases the average error induced by dithering may be compensated in a modified PSA, and the remaining error will be decorrelated to a certain extent because discontinuities will be misaligned. The expected reduction in the standard deviation of the phase error is typically less than a factor of 2 for many PSAs and therefore the strategies may be of theoretical interest only.

We believe that our sawtooth series approach can add some insight into error correlation analysis and we shall present a number of new statistical results for the phase errors in N frame PSAs.

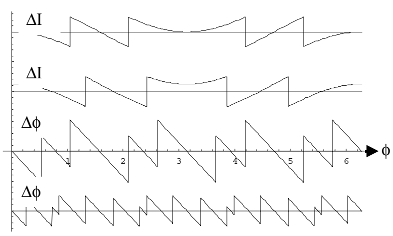

Figure 1. Intensity and phase quantisation errors

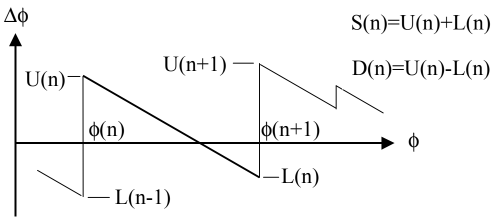

Figure 2. Phase error parameters

1 C. Brophy, "Effect of intensity error correlation on the computed phase of phase-shifting interferometry," Journal of the Optical Society of America, A **7**, 537-541, (1990).

2 B. Zhao, "A statistical method for fringe intensity-correlated error in phase-shifting measurement: the effect of quantization error on the N-bucket algorithm," Meas. Sci. technol. **8**, 147-153, (1997).

3 K. G. Larkin, and B. F. Oreb, "Propagation of errors in different phase-shifting algorithms: a special property of the arctangent function," SPIE International Symposium on Optical Applied Science and Engineering, Vol. 1755, San Diego, California, (1992), 219-227.