Focal shift, optical transfer function, and phase-space representations

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The focal shift for a lens of finite value of Fresnel number can be defined in terms of the second moment of the intensity distribution in transverse planes. The connection with the optical transfer function is described. The specification of the focused amplitude in terms of the fractional Fourier transform is discussed, and the connections among the fractional Fourier transform, the Wigner distribution, and the ambiguity function are described, leading to a model for effects of Fresnel number in terms of a rotation in phase space. The uncertainty principle is discussed, including the significance of the beam propagation factor M^2 and the width of optical fiber beam modes. Calculation of the moments in terms of the modulus and the phase of the illuminating wave is presented, and the use of the Kaiser–Teager energy operator is also described. © 2000 Optical Society of America [S0740-3232(00)00404-X]

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1. INTRODUCTION

In a recent paper De Nicola *et al.* studied the focal shift of focused beams by examining the second moment of the intensity as a function of axial position.[1] They derived an explicit exact expression for the focal shift defined in this way, which is not in general possible if focal shift is defined in terms of the axial maximum in intensity.[2] In fact, their expression is essentially identical with one given many years ago in a paper by Papoulis.[3] The latter paper does not seem to be widely referenced in the focal shift literature. It seems that workers in various areas such as focal shift, transfer functions, fractional Fourier transforms (FrFT's), phase-space representations, beam propagation, optical fibers, and signal processing are unaware of developments in related fields, as is evident from references in the literature. An aim of the present paper is to bring together results in these different fields with particular reference to the application of focal shift. Expressions for the width of the focused beam are presented, and expressions for the moments in terms of the modulus and the phase of the illuminating wave are given.

The focal shift can be considered in terms of the optical transfer function (OTF). It is interesting to note that the second moment of intensity has a special significance in terms of the Fourier transform of the intensity, which is formally identical with the OTF. For a focused cylindrical field [one-dimensional (1-D) aperture], the amplitude in the focal region at a point (x, z) measured from the geometrical focus, according to paraxial Fresnel theory, is

$$U(x, z) = \frac{1}{\sqrt{\lambda(f+z)}} \exp\left[ik\left[f + z + \frac{x^2}{2(f+z)}\right] - \frac{i\pi}{4}\right] \times \int_{-a}^{+a} U(x', -f) \times \exp\left[-\frac{ikx'^2}{2f} + \frac{ikx'^2}{2(f+z)} - \frac{ikxx'}{f+z}\right] \mathrm{d}x', \tag{1}$$

where $2a$ is the width of the aperture and $k = 2\pi/\lambda$. Introducing the normalized coordinate

$$\xi' = \frac{x'}{a}, \tag{2}$$

we can write the integral in a simple form by also introducing optical coordinates[4]

$$v = kx\frac{a}{f+z}, \qquad u = kz\frac{a^2}{f(f+z)}. \tag{3}$$

If we define the angular semiaperture of the system as measured from the geometrical focus as α_0 and that as measured from the point of observation as $\alpha(z)$, the amplitude is then (assuming that α is small, so that $\sin\alpha \approx \tan\alpha$)

$$U(v, u) = \left\{\frac{a}{\lambda}\sin[\alpha(z)]\right\}^{1/2} \exp[ik(f+z)]\exp\left(-\frac{i\pi}{4}\right) \times \exp\left[\frac{iv^2}{2ka\sin[\alpha(z)]}\right] \times \int_{-1}^{1} U(\xi', -f)\exp(-iv\xi') \times \exp\left(-\frac{iu\xi'^2}{2}\right)\mathrm{d}\xi'. \tag{4}$$

We see that Eqs. (3) can also be rewritten in the form

$$v \approx kx\sin[\alpha(z)], \qquad u \approx kz\sin\alpha_0\sin[\alpha(z)], \tag{5}$$

so that both optical coordinates are scaled by the effective aperture of the system at the point of observation. In addition, the only premultiplying factor in Eq. (4) that affects the intensity is also a function of the effective aperture. We conclude that the focal shift effect can be completely described by the variation of the effective aperture with observation point. This explanation was given previously[5] when the effect of the observation point on the OTF was described. The value of the OTF at zero spatial frequency remains constant with axial position, as it must because of conservation of energy, but the spatial-frequency cutoff becomes greater closer to the aperture. The overall effect is a balance between this increase in bandwidth and defocus, the maximum on-axis intensity occurring when the integral under the OTF is a maximum.

The case of a two-dimensional (2-D) aperture follows directly, and for an axially symmetric system, Eq. (4) becomes

$$U(v,u) = \frac{ia}{\lambda}\sin[\alpha(z)]\exp[ik(f+z)]\exp\left[\frac{iv^2}{2ka\sin[\alpha(z)]}\right] \times \int_0^1 U(\rho, -f)J_0(v\rho)\exp\left(\frac{iu\rho^2}{2}\right)\rho\,\mathrm{d}\rho, \qquad (6)$$

where v and ρ are now the normalized cylindrical radii in the observation and aperture planes, respectively.

The second moment of the intensity is proportional to the second derivative at the origin of the OTF.[6] Thus this gives an alternative approach for calculation of the position of the axial minimum of the second moment of the intensity. A point that was not made by De Nicola *et al.*[1] is that for a uniformly illuminated unobstructed aperture the second moment of intensity of the focused spot is infinite, corresponding to a cusp at the origin of the OTF. However, this case can be treated as the limiting behavior of weak apodization.[7–9]

The paper is organized as follows: Section 2 develops the fractional Fourier formalism for free-space diffraction and shows that a focal shift expression can result purely from the elimination of a certain phase factor. Section 3 then considers the rotation in phase space that is an intrinsic part of the fractional Fourier formalism. Expressions for moments of a propagating beam in terms of projections and slices in phase space are derived. Section 4 derives general expressions for the focal shift and the beam M^2 factor in terms of both the phase-space rotation and the phase-space moments. Section 5 relates the beam moments to an invariant uncertainty measure. Section 6 derives a number of previously unrecorded interrelationships between the optical beam moments and width measures used in signal analysis and fiber optics. Ultimately, a simple relationship with the Kaiser–Teager energy operator is revealed. Section 7 is a discussion of the overall context of the work and of definitions of beam width.

2. EXPRESSION IN TERMS OF FRACTIONAL FOURIER TRANSFORMS

It is interesting to consider the amplitude in the focal region in terms of the FrFT.[10] Although several papers have considered free-space diffraction in terms of the FrFT,[11–16] none of these has considered specifically the case of a focused apertured beam of finite Fresnel number. We rewrite Eq. (4) in terms of the Fresnel number $N(z)$ in the observation plane,

$$N(z) = \frac{a^2}{\lambda(f+z)}, \qquad (7)$$

giving, for the 1-D aperture,

$$U(v, u) = \sqrt{N(z)}\exp[ik(f+z)]\exp\left(-\frac{i\pi}{4}\right)\exp\left[\frac{iv^2}{4\pi N(z)}\right] \times \int_{-1}^{1} U(\xi', -f)\exp(-iv\xi') \times \exp\left(-\frac{iu\xi'^2}{2}\right)\mathrm{d}\xi'. \qquad (8)$$

The FrFT is given by

$$F^{\beta}U(p) = \frac{1}{(2\pi\sin\beta)^{1/2}} \times \exp\left(-\frac{i\pi}{4}+\frac{i\beta}{2}\right)\exp\left(\frac{ip^2\cot\beta}{2}\right) \times \int_{-\infty}^{+\infty} U(p')\exp\left(\frac{ip'^2\cot\beta}{2}-\frac{ipp'}{\sin\beta}\right)\mathrm{d}p'. \qquad (9)$$

Comparing the arguments in the integrals of Eqs. (8) and (9), we have, assuming that the amplitude $U(p')$ is zero outside the aperture,

$$p'^2\cot\beta = -u\xi'^2, \qquad \frac{pp'}{\sin\beta} = v\xi'. \qquad (10)$$

As the aperture has a normalized width from -1 to $+1$, it seems sensible to take $p' = \xi'$. It should be pointed out that this choice is arbitrary, however, as is also the incorporation of an arbitrary parabolic phase variation in $U(\xi', -f)$.[15]

We then obtain

$$\cot\beta = -u, \qquad \sin\beta = \frac{1}{\sqrt{1+u^2}}, \qquad p = \frac{v}{\sqrt{1+u^2}}. \qquad (11)$$

Thus we have

$$U_u(v) = \sqrt{2\pi N(z)}(1+u^2)^{1/4} \times \exp[ik(f+z)]\exp\left(\frac{i}{2}\cot^{-1}u\right) \times \exp\left\{\frac{iv^2}{2}\left[\frac{1}{2\pi N(z)}+\frac{u}{1+u^2}\right]\right\} \times F^{-\cot^{-1}u}U\left(\frac{v}{\sqrt{1+u^2}}\right). \qquad (12)$$

The fractional order of the transformation is $2\beta/\pi$, so that in the geometric plane of focus it becomes unity, for any value of Fresnel number. Defining the Fresnel number in the geometric plane of focus as N_0, we illustrate the variation of fractional order with axial position in Fig. 1. We can then write the following for the amplitude:

$$\begin{aligned} U_u(v) = {} & \sqrt{2\pi N_0 - u}(1 + u^2)^{1/4} \\ & \times \exp[ik(f+z)]\exp\left(\frac{i}{2}\cot^{-1}u\right) \\ & \times \left\{\exp\left[\frac{iv^2(1+2\pi N_0 u)}{2(2\pi N_0 - u)(1+u^2)}\right]\right\} \\ & \times F^{-\cot^{-1}u}U\left(\frac{v}{\sqrt{1+u^2}}\right). \end{aligned} \tag{13}$$

This equation includes a phase factor given by the third complex exponential factor. This phase variation vanishes if

$$\frac{z}{f} = -\frac{1}{1 + 4\pi^2 {N_0}^2} \tag{14}$$

which is a particularly simple expression resulting from the use of the FrFT. However, there is in general an additional phase variation in the FrFT. For the particular case of a Gaussian beam, the phase of the FrFT vanishes, so that Eq. (14) defines the waist of the beam.[17]

The intensity at axial position u is in general

$$\begin{aligned} I_u(v) = {} & (2\pi N_0 - u)(1+u^2)^{1/2} \\ & \times \left|F^{-\cot^{-1}u}U\left(\frac{v}{\sqrt{1+u^2}}\right)\right|^2. \end{aligned} \tag{15}$$

If the Fresnel number N_0 of the system becomes large, corresponding to the Debye approximation, the amplitude is

$$\begin{aligned} U_u(v) = {} & \sqrt{2\pi N_0}(1+u^2)^{1/4} \\ & \times \exp[ik(f+z)]\exp\left(\frac{i}{2}\cot^{-1}u\right) \\ & \times \exp\left(\frac{iv^2 u}{2(1+u^2)}\right)F^{-\cot^{-1}u}U\left(\frac{v}{\sqrt{1+u^2}}\right). \end{aligned} \tag{16}$$

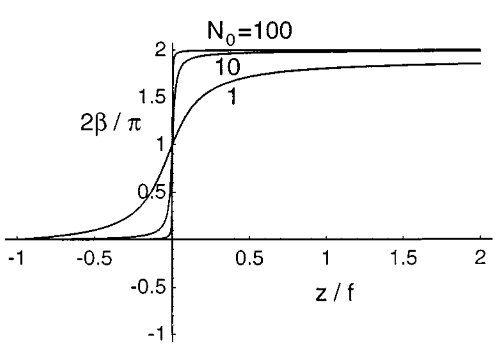

Fig. 1. Order of fractional Fourier transformation for a slit aperture illuminated by a converging wave for different values of the Fresnel number N_0.

In this case, if the original illuminating field is a real and even function, its Fourier transform is also real, and the waist of the beam corresponds to the geometric focal plane.

3. WIGNER DISTRIBUTION AND AMBIGUITY FUNCTION

It has been shown how the FrFT has a simple interpretation as a rotation in phase space.[18–21] The squared magnitude of the FrFT is given by the projection of the Wigner distribution onto a line at a particular angle. As the ambiguity function[3] is the 2-D Fourier transform of the Wigner distribution,[22,23] the squared magnitude of the FrFT is given by a section through the ambiguity function.[20] The moments of the intensity in phase space can be expressed simply by considering the Wigner distribution or the ambiguity function.

The Fourier transform of the amplitude is

$$\tilde{U}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\xi')\exp(-ip\xi')\mathrm{d}\xi'. \tag{17}$$

The Wigner distribution function $W(\xi,p)$ of the amplitude is given by

$$\begin{aligned} W(\xi, p) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\xi + \xi'/2)U^*(\xi - \xi'/2) \\ &\quad \times \exp(-ip\xi')\mathrm{d}\xi' \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{U}(p + p'/2)\tilde{U}^*(p - p'/2) \\ &\quad \times \exp(ip'\xi)\mathrm{d}p'. \end{aligned} \tag{18}$$

The Wigner distribution is a real (possibly negative) quantity. Its zero-order moments are

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} W(\xi, p)\mathrm{d}p = |U(\xi)|^2,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} W(\xi, p)d\xi = |\tilde{U}(p)|^2,$$

$$\frac{1}{\sqrt{2\pi}}\int\int_{-\infty}^{\infty} W(\xi, p)\mathrm{d}p\,\mathrm{d}\xi = E, \tag{19}$$

where E multiplied by a is the total energy, as ξ is a normalized coordinate [Eq. (2)]. The first-order moments of the Wigner distribution, which represent the position of the center of gravity of the Wigner distribution in phase space, are

$$m_\xi = \frac{1}{E}\int |U(\xi)|^2\xi\,\mathrm{d}\xi,$$

$$m_p = \frac{1}{E}\int |\tilde{U}(p)|^2 p\,\mathrm{d}p, \tag{20}$$

where from now on the limits of integration are taken to go from minus to plus infinity except where otherwise stated. The central second moments of the Wigner distribution are

$$m_{\xi\xi} = \frac{1}{E}\int |U(\xi)|^2(\xi - m_\xi)^2\,\mathrm{d}\xi,$$

$$m_{pp} = \frac{1}{E}\int |\tilde{U}(p)|^2(p - m_p)^2\,\mathrm{d}p. \tag{21}$$

The central mixed second-order moment, or product of inertia, about the center of gravity, is

$$m_{\xi p} = m_{p\xi} = \frac{1}{\sqrt{2\pi}E}\iint W(\xi,p)(\xi - m_\xi)(p - m_p)\mathrm{d}p\,\mathrm{d}\xi. \tag{22}$$

As the projection of the Wigner distribution onto a line through the origin of phase space at an angle β to the spatial axis, that is, the Radon transform of the Wigner distribution, is equal to the squared modulus of the FrFT,[18–21,24] the second moment of the power density of the FrFT is the second moment of the projection of the Wigner distribution. This second moment is thus identical with the second moment of the Wigner distribution about an arbitrary axis through its center of gravity:

$$m_{\xi\xi}'(\beta) = m_{\xi\xi}\cos^2\beta + m_{pp}\sin^2\beta - 2m_{p\xi}\sin\beta\cos\beta. \tag{23}$$

The locus of the radius of gyration in phase space is the inverse of the inertia ellipse,[25,26] thus giving a quartic curve, as illustrated in Fig. 2.

The ambiguity function $A(\xi, p)$ is the 2-D Fourier transform of the Wigner distribution (actually a Fourier transform in space and an inverse Fourier transform in spatial frequency):

$$A(\xi, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\xi' + \xi/2)U^*(\xi' - \xi/2) \times \exp(-ip\xi')\mathrm{d}\xi'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{U}(p' + p/2)\tilde{U}^*(p' - p/2) \times \exp(ip'\xi)\mathrm{d}p'. \tag{24}$$

In statistical terminology, if the Wigner distribution is a distribution function, then the ambiguity function is a characteristic function. The ambiguity function is Hermitian. It has the special values

$$A(\xi, 0) = \frac{1}{\sqrt{2\pi}}\int |\tilde{U}(p')|^2 \exp(ip'\xi)\mathrm{d}p',$$

$$A(0, p) = \frac{1}{\sqrt{2\pi}}\int |U(\xi')|^2 \exp(-ip\xi')\mathrm{d}\xi',$$

$$A(0, 0) = \frac{E}{\sqrt{2\pi}}. \tag{25}$$

The first moments and the central second moments of the signal are then given in terms of the ambiguity function[3]:

$$m_\xi = -\frac{\sqrt{2\pi}}{E}\left.\frac{\partial A(0, p)}{\partial p}\right|_{p=0},$$

$$m_p = -\frac{\sqrt{2\pi}}{E}\left.\frac{\partial A(\xi, 0)}{\partial \xi}\right|_{\xi=0}$$

$$m_{\xi\xi} = -\frac{\sqrt{2\pi}}{E}\left.\frac{\partial^2 A(0, p)}{\partial p^2}\right|_{p=0} - m_\xi^2,$$

$$m_{pp} = -\frac{\sqrt{2\pi}}{E}\left.\frac{\partial^2 A(\xi, 0)}{\partial \xi^2}\right|_{\xi=0} - m_p^2,$$

$$m_{\xi p} = -\frac{\sqrt{2\pi}}{E}\left.\frac{\partial^2 A(\xi, p)}{\partial\xi\partial p}\right|_{\xi=0,\,p=0} - m_\xi m_p. \tag{26}$$

4. FOCUSED BEAM

The intensity in the focal region was given in terms of the optical coordinates in Eq. (15). We can write

$$\frac{z}{f} = -\frac{1}{1 + 2\pi N_0\tan\beta} \tag{27}$$

for z, so that

$$v = \frac{x}{a}\frac{(\cos\beta + 2\pi N_0\sin\beta)}{\sin\beta}, \tag{28}$$

giving the following for the intensity:

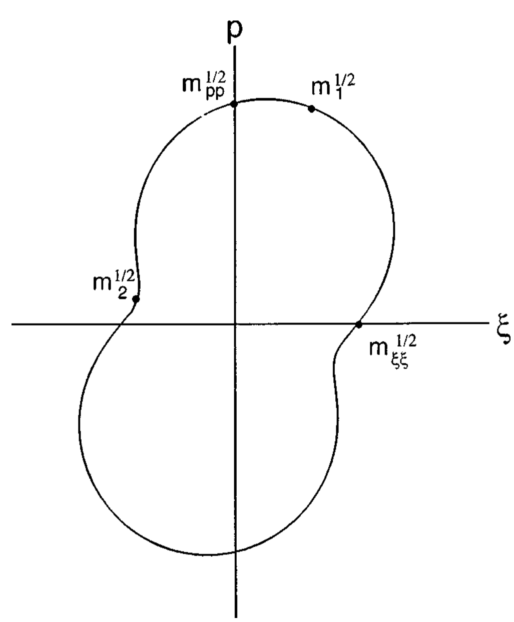

Fig. 2. Locus of the radius of gyration of the Wigner distribution function in phase space.

$$I_\beta(x) = \frac{\cos\beta + 2\pi N_0 \sin\beta}{\sin^2\beta} \times \left| F^\beta U\left(\frac{x}{a}(\cos\beta + 2\pi N_0 \sin\beta)\right)\right|^2. \quad (29)$$

The width w of the beam for a particular value of β, defined in terms of the second moment, can thus be obtained from Eq. (23) after scaling according to Eq. (29), giving

$$w^2 = a^2 \frac{m_{\xi\xi}\cos^2\beta + m_{pp}\sin^2\beta - 2m_{\xi p}\sin\beta\cos\beta}{(\cos\beta + 2\pi N_0 \sin\beta)^2}. \quad (30)$$

By differentiating with respect to $\tan\beta$, we can find that the beam waist occurs when

$$\cot\beta = \frac{m_{pp} + 2\pi N_0 m_{\xi p}}{2\pi N_0 m_{\xi\xi} + m_{\xi p}}, \quad (31)$$

which, with the use of Eq. (27), gives the following for the fractional focal shift:

$$\frac{z}{f} = -\frac{m_{pp} + 2\pi N_0 m_{\xi p}}{4\pi^2 N_0{}^2 m_{\xi\xi} + m_{pp} + 4\pi N_0 m_{\xi p}}. \quad (32)$$

At the waist we have

$$w^2 = a^2 \frac{m_{\xi\xi} m_{pp} - m_{\xi p}{}^2}{4\pi^2 N_0{}^2 m_{\xi\xi} + m_{pp} + 4\pi N_0 m_{\xi p}}. \quad (33)$$

These expressions are a generalization of those given by Papoulis,[3] and by De Nicola *et al.*[1] for the case of a 2-D aperture.

In the far field, $u \to 2\pi N_0$. The divergence of the beam, defined as $\theta = w/z$, can be written as

$$\theta^2 = \frac{4\pi^2 N_0{}^2 m_{\xi\xi} + m_{pp} + 4\pi N_0 m_{\xi p}}{4\pi^2 N_0{}^2 f^2}. \quad (34)$$

The second moments obey the Heisenberg uncertainty principle, which states that[27,28]

$$4 m_{\xi\xi} m_{pp} \geq 1, \quad (35)$$

the equality holding for the case of a Gaussian beam.

We can thus also obtain an expression for the beam propagation factor M^2,[29] which can be defined as the product of the waist radius and the far-field divergence, normalized by the corresponding values for a Gaussian beam. We obtain

$$M^2 = 2(m_{\xi\xi} m_{pp} - m_{\xi p}{}^2)^{1/2}, \quad (36)$$

which can be written as

$$M^4 = 4\,\det(\mathbf{M}), \quad (37)$$

where

$$\mathbf{M} = \begin{bmatrix} m_{\xi\xi} & m_{\xi p} \\ m_{\xi p} & m_{pp} \end{bmatrix}. \quad (38)$$

Bastiaans[27] has shown that M^2 is invariant on propagation through a first-order optical system, which can be described by an *ABCD* matrix.[30,31] Papoulis,[3] and later Bélanger,[32] have presented expressions for the beam width for arbitrary *ABCD* systems. The FrFT associated with an *ABCD* transformation has also been discussed.[33]

On fractional Fourier transformation, the moment tensor $\mathbf{M}$ is transformed to

$$\mathbf{M}' = \mathbf{R}^{\mathrm{T}}\mathbf{M}\mathbf{R}, \quad (39)$$

where $\mathbf{R}$ is the rotation matrix

$$\mathbf{R} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}. \quad (40)$$

Thus, in addition to Eq. (23), which gives the second moment $m'_{\xi\xi}$, we also have

$$\begin{aligned} m'_\xi &= m_\xi \cos\beta + m_p \sin\beta, \\ m'_p &= m_p \cos\beta - m_\xi \sin\beta, \\ m'_{pp} &= m_{\xi\xi}\sin^2\beta + m_{pp}\cos^2\beta \\ &\quad + 2m_{\xi p}\sin\beta\cos\beta, \\ m'_{\xi p} &= (m_{\xi\xi} - m_{pp})\sin\beta\cos\beta \\ &\quad + m_{\xi p}\cos 2\beta. \end{aligned} \quad (41)$$

These expressions are analogous to those given elsewhere for the propagation of moments.[3,27,32] However, on propagation, there is also the scaling in Eq. (28), so that m'_p, m'_{pp} scale inversely to $m_\xi, m_{\xi\xi}$, while $m'_{\xi p}$ does not scale. The position of the center of gravity x_0 as a function of β can be determined from Eqs. (28) and (41) as

$$x_0 = a\frac{m_\xi \cos\beta + m_p \sin\beta}{\cos\beta + 2\pi N_0 \sin\beta}. \quad (42)$$

The center of gravity of the beam is seen to propagate in a straight line, as is well-known.

5. UNCERTAINTY RELATIONSHIPS

The Heisenberg uncertainty relationship in expression (35) is invariant with scaling but is variant with fractional Fourier transformation (rotation). On the other hand, the polar second moment m_W, given by

$$m_W = m_{\xi\xi} + m_{pp}, \quad (43)$$

is invariant on rotation, by the perpendicular axes theorem, but is scale variant. The principal second moments are obtained as the eigenvalues of $\mathbf{M}$, giving

$$m_{1,2} = \tfrac{1}{2}(m_W \pm \sqrt{m_W{}^2 - M^4}). \quad (44)$$

The product of the principal second moments is

$$m_1 m_2 = M^4/4. \quad (45)$$

This is invariant with both rotation and scaling, as has been pointed out previously.[27,28]

6. RELATIONSHIPS BETWEEN THE MOMENTS

Not every function is a Wigner function, as it must satisfy the Fourier transform relationship. In fact, all the moments can be determined from a knowledge of $U(\xi)$. From Eqs. (20) we have the following for the mean value m_p:

$$m_p = \frac{i}{2E}\int\left[\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi}U^*(\xi) - U(\xi)\frac{\mathrm{d}U^*(\xi)}{\mathrm{d}\xi}\right]\mathrm{d}\xi. \quad (46)$$

Writing $U(\xi)$ in terms of its modulus and phase,

$$U(\xi) = a(\xi)\exp[i\theta(\xi)], \quad (47)$$

and expanding the derivative, we then have

$$m_p = -\frac{1}{E}\int a^2(\xi)\frac{\mathrm{d}\theta(\xi)}{\mathrm{d}\xi}\mathrm{d}\xi, \quad (48)$$

that is, the mean spatial frequency is proportional to the weighted phase derivative.[34]

Using Parseval's theorem, we obtain the following for the central second moment in spatial frequency:

$$m_{pp} = \frac{1}{E}\int\left|\frac{\mathrm{d}}{\mathrm{d}\xi}[U(\xi)\exp(im_p\xi)]\right|^2\mathrm{d}\xi. \quad (49)$$

This expression was given by Pohlig.[35] It was later given by Bélanger[32] in connection with *ABCD* systems for the special case when m_p is zero. It is also well-known in its circularly symmetric form as the so-called Petermann 2 width[36] of optical fiber beam modes, the significance of which was explained by Pask.[37] This has also been generalized to the noncircularly symmetric case.[38]

Writing $U(\xi)$ in terms of its modulus and phase, we obtain[7,35]

$$m_{pp} = \frac{1}{E}\int\left\{\left[\frac{\mathrm{d}a(\xi)}{\mathrm{d}\xi}\right]^2 + a^2(\xi)\left[\frac{\mathrm{d}\theta(\xi)}{\mathrm{d}\xi} + m_p\right]^2\right\}\mathrm{d}\xi. \quad (50)$$

The second moment comprises two components. In the absence of aberrations, $\tilde{U}(p)$ is Hermitian, so that m_p is zero and the second term vanishes. Thus the first term gives the contribution that would be present in the absence of aberrations. The second term, given by the weighted integral of the square of the group delay, gives the contribution that is due to the aberrations. The width is thus made up of two components, an aberration-free diffraction term and an aberration term added in quadrature. This is analogous to a method often used in electron optics.[39]

For a beam that has a hard-edged aperture at $\xi = \pm 1$, the first term can be evaluated by integration by parts,[7,40] and using Eq. (47) gives

$$m_{pp} = \frac{1}{E}\left\{\left[a(\xi)\frac{\mathrm{d}a(\xi)}{\mathrm{d}\xi}\right]_{-1}^{1} - \int_{-1}^{1}a(\xi)\frac{\mathrm{d}^2a(\xi)}{\mathrm{d}\xi^2}\mathrm{d}\xi + \int_{-1}^{1}a^2(\xi)\left[\frac{\mathrm{d}\theta(\xi)}{\mathrm{d}\xi}\right]^2\mathrm{d}\xi\right\} - m_p^2. \quad (51)$$

For a uniformly illuminated aperture, the first term becomes infinite, and the second moment is indeterminate. However, if the illumination is tapered to zero at the edges of the aperture, the first term vanishes. Papoulis[41] has shown that for a hard-edged aperture the second moment satisfies the inequality

$$m_{pp} \geqslant \frac{\pi^2}{4}, \quad (52)$$

the equality holding when

$$U(\xi) = \cos\frac{\pi\xi}{2}, \quad (53)$$

in which case

$$m_{\xi\xi} = \frac{\pi^2 - 6}{3\pi^2}, \quad (54)$$

$$M^2 = \left(\frac{\pi^2 - 6}{3}\right)^{1/2} = 1.136. \quad (55)$$

An alternative expression for the central second moment can be derived from the Wigner distribution function, giving

$$m_{pp} = \frac{1}{4E}\int\left\{2\left|\frac{\mathrm{d}U(\xi)}{d\xi}\right|^2 - \left[\frac{\mathrm{d}^2U(\xi)}{\mathrm{d}\xi^2}U^*(\xi) + U(\xi)\frac{\mathrm{d}^2U^*(\xi)}{\mathrm{d}\xi^2}\right]\right\}\mathrm{d}\xi - m_p^2, \quad (56)$$

which can be written as

$$m_{pp} = \frac{1}{2E}\int\Psi(U(\xi))\mathrm{d}\xi - m_p^2, \quad (57)$$

where $\Psi(\xi)$ is the Kaiser–Teager energy operator[42,43]

$$\Psi(U(\xi)) = \left|\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi}\right|^2 - \frac{1}{2}\left[\frac{\mathrm{d}^2U(\xi)}{\mathrm{d}\xi^2}U^*(\xi) + U(\xi)\frac{\mathrm{d}^2U^*(\xi)}{\mathrm{d}\xi^2}\right]. \quad (58)$$

The Kaiser–Teager operator, a familiar demodulator in signal and image processing, is usually defined to be a local energy measure for oscillating (simple harmonic) signals. In this instance it is applied directly to the light beam amplitude, and its meaning can be reinterpreted in relation to the second moment. In terms of modulus and phase, we then have, from Eqs. (26) and the fact that the energy operator is related to the second derivative of the ambiguity function[44] through a Fourier transformation,

$$m_{pp} = \frac{1}{2E}\int\left\{\left[\frac{\mathrm{d}a(\xi)}{\mathrm{d}\xi}\right]^2 - a(\xi)\frac{\mathrm{d}^2a(\xi)}{\mathrm{d}\xi^2} + 2a^2(\xi)\left[\frac{\mathrm{d}\theta(\xi)}{\mathrm{d}\xi}\right]^2\right\}\mathrm{d}\xi - m_p^2. \quad (59)$$

Equations (50) and (59) are equivalent for finite E, as, by integration by parts,

$$\int_{-\infty}^{\infty}\left\{a(\xi)\frac{\mathrm{d}^2a(\xi)}{\mathrm{d}\xi^2} + \left[\frac{\mathrm{d}a(\xi)}{\mathrm{d}\xi}\right]^2\right\}\mathrm{d}\xi = \left[a(\xi)\frac{\mathrm{d}a(\xi)}{\mathrm{d}\xi}\right]_{-\infty}^{\infty}. \quad (60)$$

The mixed second moment can also be evaluated in a similar way. We obtain

$$m_{\xi p} = -\frac{i}{2E}\int\left[U(\xi)\frac{\mathrm{d}U^*(\xi)}{\mathrm{d}\xi} - U^*(\xi)\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi}\right]\xi\,\mathrm{d}\xi - m_p m_\xi, \tag{61}$$

or

$$m_{\xi p} = -\frac{1}{E}\int a^2(\xi)\frac{\mathrm{d}\theta(\xi)}{\mathrm{d}\xi}\xi\,\mathrm{d}\xi - m_p m_\xi, \tag{62}$$

in which the first term is proportional to the weighted first moment of the phase derivative. If $U(\xi)$ is real or Hermitian, the mixed second moment vanishes.[3,32]

7. DISCUSSION

The propagation of moments of an optical beam focused by a lens has been considered. These can be calculated from the illuminating field by using the Wigner distribution or the ambiguity function. The transformation to the Wigner distribution on propagation can be considered in terms of the fractional Fourier transform, resulting in a rotation and a scaling. Alternatively, as discussed by Papoulis,[3] the transformation can be expressed as a shear resulting from the focusing followed by another shear from propagation. Expressions for the width of the beam as defined by the second moment have been given.

It should be noted that there are many alternative definitions of the width of a beam. The disadvantage of the second-moment definition is that it may diverge for hard-edged apertures. An alternative definition is in terms of the normalized second derivative at the peak intensity.[45] This can be readily calculated from the second moments of the ambiguity function or the second derivatives of the Wigner distribution.

The interconnections among focal shift, optical transfer functions, beam width, signal duration, phase-space moments, the Heisenberg uncertainty, and the Kaiser–Teager energy operator revealed in this investigation may be used to reinterpret the concept of focal shift.

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