Wigner function for nonparaxial wave fields

Colin J. R. Sheppard* **and Kieran G. Larkin†**

Department of Physical Optics, School of Physics, University of Sydney, Sydney, NSW 2006, Australia

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The generalized optical transfer function and the spectral correlation function are investigated for nonparaxial two-dimensional wave fields. The angle-impact marginal of the four-dimensional Wigner function is derived directly. For focused wave fields of semiangle greater than 90°, the spectral correlation function exhibits overlapping and interference. For focused wave fields for which the semiangle is known to be less than 180°, the magnitude and phase can be recovered directly from knowledge of the intensity in the focal region. © 2001 Optical Society of America

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1. INTRODUCTION

Recently, Wolf *et al.*[1] investigated the Wigner function for high-aperture two-dimensional (2-D) wave fields (i.e., fields that propagate in the $x-z$ plane such that the field is constant in the y direction perpendicular to the plane) that satisfy the 2-D scalar Helmholtz equation. They showed that the conventionally defined 2-D Wigner function, unlike that for the paraxial case, is not propagation invariant. They described a four-dimensional (4-D) Wigner function and derived from it what they termed a 2-D angle-impact marginal from which the (complex) spectral function (the same as the angular spectrum, or angular pupil function for a focused field in the Debye approximation) can be recovered. They did not address the problem of calculating the angle-impact marginal from the focused field intensity.

We recently demonstrated[2] a direct (noniterative) phase-retrieval method for 2-D high-aperture wave fields, based on the concept of the generalized optical transfer function (OTF). The 2-D generalized OTF, first described by Mertz for the short-wavelength limit,[3] is the 2-D Fourier transform of the intensity. The term "generalized" is used to distinguish this OTF from the ordinary defocused OFT, which is the one-dimensional Fourier transform (in x) of the intensity at a fixed value of z. The region of support for the 2-D generalized OTF for the high-angle case was given by Sheppard,[4] and its value for various cases within that region of support has also been presented.[5] The corresponding high-angle three-dimensional case has also been derived;[6] the notation is similar to that adopted here.

Our aim in this paper is to explore the relationship between the 2-D generalized OTF and the Wigner function for high-angle 2-D scalar wave fields. In particular the angle-impact marginal is derived directly, without introducing the 4-D Wigner function, which contains redundant information. We also show that phase of the spectral function, and therefore of the focused field, can be recovered from the 2-D intensity information even in the presence of backward-propagating components.

2. CONVERGENT SCALAR WAVE SPACE FIELDS

Consider a convergent quasi-monochromatic angular spectrum (pupil function) of strength $P(\theta)$ in free space. There are no evanescent components. Then the amplitude in the focal region is evaluated by integrating over the angular spectrum[7] (Fig. 1):

$$U(x, z) = \int_{-\pi}^{\pi} P(\theta)\exp[ik(x\sin\theta - z\cos\theta)]\mathrm{d}\theta, \quad (1)$$

where $k = 2\pi/\lambda$. The amplitude can also be written[8] in terms of the generalized 2-D pupil function $\Pi(m, s)$:

$$U(x, z) = \frac{k}{2\pi}\int\int \Pi(m, s)\exp[ik(mx + sz)]\mathrm{d}m\mathrm{d}s, \quad (2)$$

where m and s are normalized spatial frequencies in the x and z directions. Limits of integration are taken as $\pm\infty$ unless otherwise stated. As the field satisfies the Helmoltz equation, the normalized spatial frequencies (direction cosines) m and s, given by

$$m = K\sin\theta,$$
$$s = -K\cos\theta, \quad (3)$$

are constrained to lie on the Ewald circle (the 2-D form of the Ewald sphere), so that

$$\Pi(m, s) = \frac{2\pi}{k}P(\theta)\delta(K - 1). \quad (4)$$

Equation (2) represents a 2-D Fourier transform, which can be evaluated by transformation into polar coordinates K, θ by use of Eq. (4). Then, after the integral in K, is evaluated, Eq. (1) results. Equation (2) can be inverted to give

$$\Pi(m, s) = \frac{k}{2\pi}\int\int U(x, z)\exp[-ik(mx + sz)]\mathrm{d}x\mathrm{d}z. \quad (5)$$

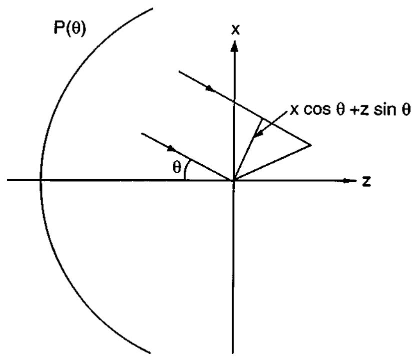


Fig. 1. Field at a point in x, z space produced from an angular spectrum (pupil function) $P(\theta)$.

Thus the generalized 2-D pupil function is the 2-D Fourier transform of the amplitude, in the same way that the 2-D generalized OTF is the 2-D Fourier transform of the intensity.

3. FIELD INTENSITY AND GENERALIZED OPTICAL TRANSFER FUNCTION

We now establish analogous expressions for the intensity of the field. The 2-D generalized OTF is given by the 2-D Fourier transform of the intensity:

$$G(m,\ s) = \frac{k}{2\pi}\int\int |U(x,\ z)|^2 \exp[-ik(mx + sz)]\mathrm{d}x\mathrm{d}z, \tag{6}$$

which, by the convolution theorem, gives

$$G(m,\ s) = \frac{k}{2\pi}\int\int \Pi(m' + m/2, s' + s/2)\Pi^*(m' - m/2, s' - s/2)\mathrm{d}m'\mathrm{d}s'. \tag{7}$$

The 2-D generalized OTF is thus given[9] by the autocorrelation of the 2-D generalized pupil function. To calculate this OTF we must take into account the finite spread of frequencies, so that the Ewald circles are in fact rings. Thus the 2-D generalized OTF is given by the overlap of two displaced rings, and overlap depends on the angle of intersection of the rings.[2,5,6] In terms of the notation of Fig. 2 we then have, for values of m and s within the region of support,

$$G(m,\ s) = \frac{2\pi}{k}\frac{P(\theta_1)P^*(\theta_2)}{|\sin(\theta_1 - \theta_2)|} = \frac{P(\theta' + \alpha/2)P^*(\theta' - \alpha/2)}{|\sin\alpha|}, \tag{8}$$

where

$$K = 2\sin(\alpha/2), \tag{9}$$

so that

$$m = 2\sin\frac{\alpha}{2}\cos\theta', \qquad s = 2\sin\frac{\alpha}{2}\sin\theta', \tag{10}$$

$$|\sin\alpha| = |K|(1 - K^2/4)^{1/2}. \tag{11}$$

The two rings intersect in general at two points, corresponding to the two roots of

$$s/m = \tan\theta', \tag{12}$$

which represent forward- and backward-propagating waves. In general, $G(m,\ s)$ is given by the sum of two terms of the form given in Eq. (8). The function $G(m,\ s)$ is Hermitian.

In analogy with Papoulis,[10] we introduce the function

$$\gamma(\alpha,\ \theta') = P(\theta' + \alpha/2)P^*(\theta' - \alpha/2), \tag{13}$$

which we call the spectral correlation function. Note, however, that the arguments in Eq. (13) are angles, rather than sines of angles as in the Papoulis representation. The pupils are rotated relative to each other. This property is a consequence of the geometry of the triangle in Fig. 2. The two terms of $G(m,\ s)$ are given by two values of θ' in the range $-\pi \leqslant \theta' \leqslant \pi$. This is equivalent to two values of α, which can be written as α, $\pi-\alpha$. We then have the relationship

$$G(m,\ s) = \frac{2\pi}{k}\frac{\gamma\left[2\arcsin\left(\frac{m^2+s^2}{2}\right), \arctan\left(\frac{s}{m}\right)\right] + \gamma\left[2\pi - 2\arcsin\left(\frac{m^2+s^2}{2}\right), \arctan\left(\frac{s}{m}\right)\right]}{(m^2+s^2)^{1/2}\left(1 - \frac{m^2+s^2}{4}\right)^{1/2}}, \tag{14}$$

and, for the special case of forward-propagating waves only,

$$\gamma(\alpha,\ \theta') = \frac{k}{2\pi}|\sin\alpha| G\left(2\sin\frac{\alpha}{2}\cos\theta',\ 2\sin\frac{\alpha}{2}\sin\theta'\right). \tag{15}$$

We can also introduce the variable

$$m' = \left(1 - \frac{K^2}{4}\right)^{1/2}\sin\theta', \tag{16}$$

giving

$$m' = \frac{s}{K}\left(1 - \frac{K^2}{4}\right)^{1/2}, \tag{17}$$

which reduces to the usual paraxial form $s/m = m'$ in the paraxial limit when m' is small.[9]

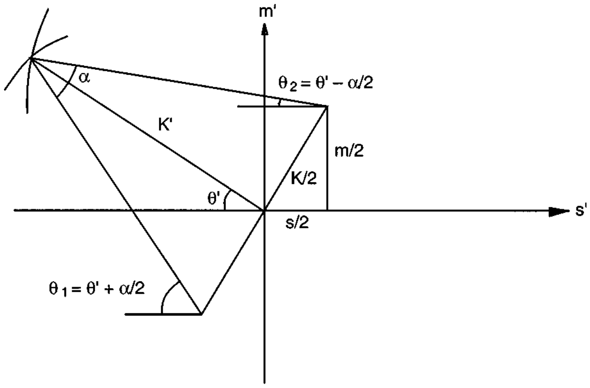


Fig. 2. Generalized OTF calculated from the overlap of two circles and centered at $m/2$, $s/2$ and $-m/2$, $-s/2$.

4. WIGNER FUNCTION

Equation (6) can be inverted to give

$$|U(x, z)|^2 = \frac{k}{2\pi}\int\int G(m, s)\exp[ik(mx + sz)]\mathrm{d}m\mathrm{d}s, \tag{18}$$

where the integral is taken over the region where $G(m, s)$ is nonzero, which is in general inside a circle of radius equal to 2. We can write the intensity in terms of the spectral correlation function by expressing $G(m, s)$ in terms of the spectral correlation function [Eq. (14)], transforming to polar coordinates, and introducing the variable

$$l = x\cos\theta' + z\sin\theta'. \tag{19}$$

Alternatively, starting directly from Eq. (1),

$$\begin{aligned}|U(x, z)|^2 = \int\int\ & P(\theta_1)\exp[ik(x\sin\theta_1 - z\cos\theta_1)] \\ & \times P^*(\theta_2)\exp[-ik(x\sin\theta_2 \\ & - z\cos\theta_2)]\mathrm{d}\theta_1\mathrm{d}\theta_2.\end{aligned} \tag{20}$$

We transform the integration variables to θ', α: The transformation is illustrated in Fig. 3. For an optical field of small semiangle, the support repeats periodically. The rectangular region inclined at 45° (Fig. 4) contains the same information as the square θ_1, θ_2 region but rearranged. We see that the two points of intersection of the generalized pupils are accounted for by taking both negative and positive values of the variable α. Thus

$$\begin{aligned}|U(x, z)|^2 = & \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\gamma(\alpha, \theta') \\ & \times \exp\left[2ik\sin\frac{\alpha}{2}(x\cos\theta' + z\sin\theta')\right]\mathrm{d}\theta'\mathrm{d}\alpha.\end{aligned} \tag{21}$$

Equation (21) is a double integral, which we can perform by evaluating the integral in α first to get

$$M(\theta', l) = \left(\frac{k}{2\pi}\right)^{1/2}\int_{-\pi}^{\pi}\gamma(\alpha, \theta')\exp\left(2ikl\sin\frac{\alpha}{2}\right)\mathrm{d}\alpha, \tag{22}$$

so that

$$|U(x, z)|^2 = \left(\frac{2\pi}{k}\right)^{1/2}\int M(\theta', x\cos\theta' + z\sin\theta')\mathrm{d}\theta'. \tag{23}$$

Equation (22) defines the angle-impact marginal of Wolf *et al.*,[1] which we call the Wigner function for simplicity. In the paraxial limit it reduces to the ordinary Wigner function $W(m', x)$. This is a novel derivation that avoids introduction of 4-D functions.

Inverting Eq. (22) yields

$$\gamma(\alpha, \theta') = \left(\frac{k}{2\pi}\right)^{1/2}\cos\frac{\alpha}{2}\int_{-\infty}^{\infty}M(\theta', l)\exp\left(-2ikl\sin\frac{\alpha}{2}\right)\mathrm{d}l. \tag{24}$$

We note that, for $l = 0$, Eq. (22) reduces to the circular autocorrelation[11] of the pupil function. Putting $\alpha = 0$ yields

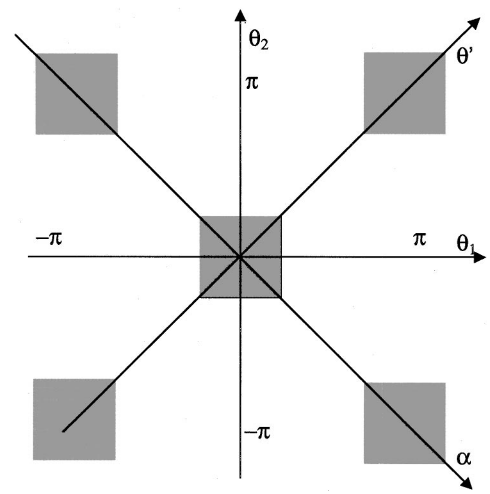


Fig. 3. Transformation of the integration variables from θ_1, θ_2 to θ', α. For an optical field of small semiangle, the support repeats periodically.

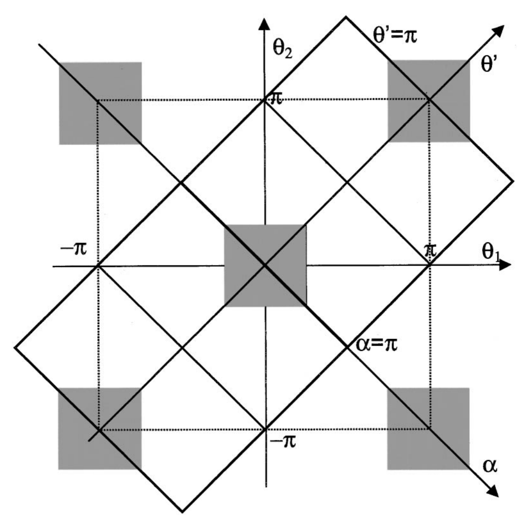


Fig. 4. The rectangular region inclined at 45° contains the same information as the square θ_1, θ_2 region but rearranged.

$$|P(\theta')|^2 = \left(\frac{k}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} M(\theta', l)\mathrm{d}l. \tag{25}$$

Alternatively, putting θ_1 or θ_2 constant in Eq. (24) allows the angular spectrum to be recovered from the Wigner function to within a constant phase factor. We also have

$$\int |P(\theta')|^2 \mathrm{d}\theta = \frac{k}{2\pi} \int |U(x, z)|^2 \mathrm{d}l = \left(\frac{k}{2\pi}\right)^{1/2} \int\int M(\theta', l)\mathrm{d}l\mathrm{d}\theta = E, \tag{26}$$

which is the total energy, a constant that is invariant under translation or rotation.

For the case of forward-propagating waves only, by using Eq. (15) we can obtain an explicit Fourier relationship between the generalized OTF and the Wigner function:

$$G(m, s) = \left(\frac{2\pi}{k}\right)^{1/2} \frac{1}{(m^2 + s^2)^{1/2}} \int_{-\infty}^{\infty} M\left[\arctan\left(\frac{s}{m}\right), l\right] \times \exp(-iklK)\mathrm{d}l. \tag{27}$$

For a wave field consisting of forward- and backward-propagating components, it is not in general possible to obtain this relationship, as can be seen from Eq. (14).

5. DEFOCUSED OPTICAL TRANSFER FUNCTION

The defocused OTF for a forward-propagating wave is given by the one-dimensional Fourier transform of the 2-D generalized OTF:

$$C(m, z) = \left(\frac{k}{2\pi}\right)^{1/2} \int G(m, s)\exp(iksz)\mathrm{d}s, \tag{28}$$

which can be written in the alternative forms

$$C(m, z) = \left(\frac{2\pi}{k}\right)^{1/2} \int \frac{\gamma(\alpha, \theta')}{(\cos^2\theta' - m^2/4)^{1/2}} \exp(ikmz\tan\theta')\mathrm{d}\theta' \tag{29}$$

$$= \left(\frac{2\pi}{k}\right)^{1/2} \int \frac{\gamma(\alpha, \theta')}{(1 - K^2/4)^{1/2}(K^2 - m^2)^{1/2}} \times \exp[ikz(K^2 - m^2)^{1/2}]\mathrm{d}K. \tag{30}$$

For $m = 0$ we then have

$$C(0, z) = \left(\frac{2\pi}{k}\right)^{1/2} \int_{-\pi}^{\pi} \frac{|P(\theta')|^2}{\cos\theta'} \mathrm{d}\theta' \tag{31}$$

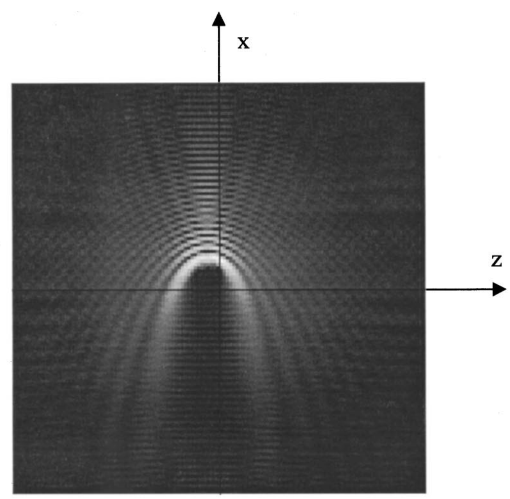
for all z, which represents the power flow in the z direction and allows normalization of the OTF.

6. DISCUSSION

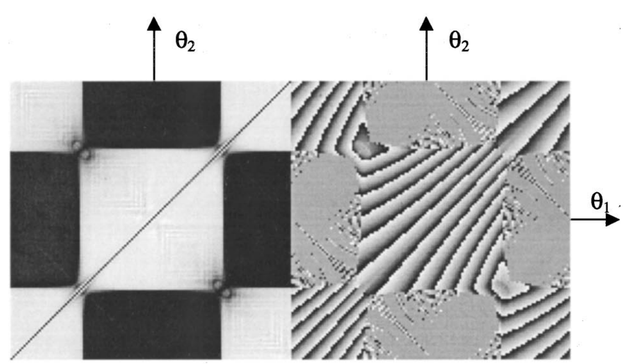
We have derived the angle-impact marginal without introducing the 4-D Wigner function. Relationships have also been given for the generalized OTF and the spectral correlation function. Knowledge of the Wigner function allows the field and the angular spectrum to be recovered. This is the basis of the phase-retrieval method based on Wigner functions.[12,13] However, in general for high-aperture fields there is an ambiguity caused by the presence of forward- and backward-propagating waves. For example, it is obviously impossible to distinguish from the intensity in the focal region whether the intensity was produced by forward- or backward-propagating waves. Thus the Wigner function also cannot in general be determined from knowlege of the focal intensity.

In the special case when the wavelength is known and the pupil semiangle is known to be less than π, there are regions of the spectral correlation function $P(\theta_1)P^*(\theta_2)$ that do not overlap [diagonal regions of $\gamma(\theta', \alpha)$], so the original phase and amplitude $P(\theta_1)$ can be recovered unambiguously.

Figure 5 shows an example of an intensity plot for a field that has a semiangle equal to $\pi/2$. This corresponds to the limiting case when the spectral correlation func-

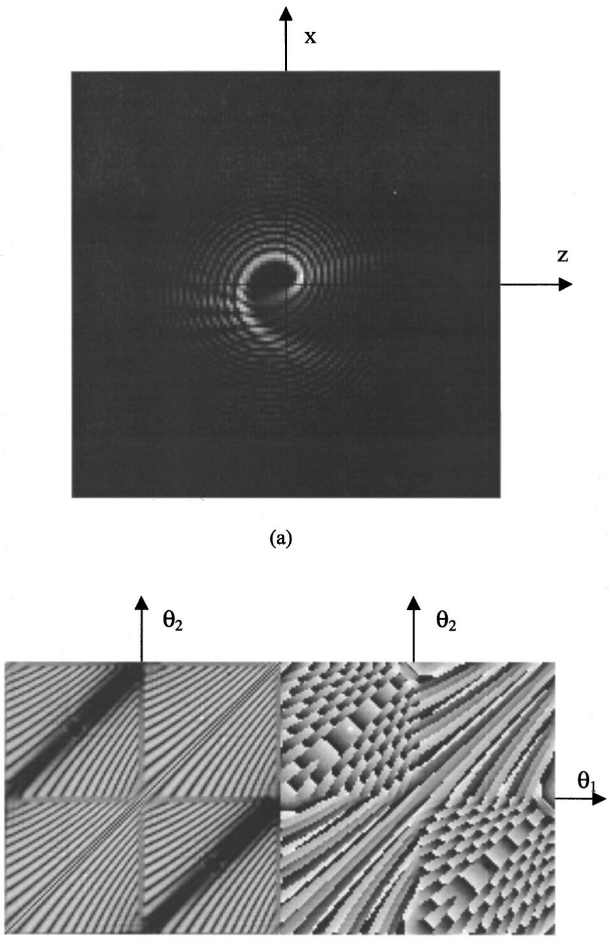


(a)



(b)

Fig. 5. (a) Example of an intensity plot for a field with a chirped phase that has a semiangle equal to $\pi/2$. This corresponds to the limiting case for spectral correlation function overlap. (b) Magnitude (left) and phase (right) of the spectral correlation function extracted from the intensity map by using the algorithm developed in a previous paper.[2] The magnitude component clearly shows that the functions do not overlap but just touch at the corners.



(b)

Fig. 6. (a) Intensity plot for a field with a chirped phase that has a semiangle almost equal to π, in other words, almost the full circle. (b) Magnitude (left) and phase (right) of the spectral correlation function again extracted from the intensity map by using the algorithm developed in a previous paper.[2] The magnitude component clearly shows almost complete overlap and interference, except in an x–y cross pattern at the center. The actual position of the cross depends on the location of the gap in the radiation direction.

tions just do not overlap. The spectral correlation function in Fig. 5(b) is extracted directly from the intensity map by use of an algorithm developed in a previous paper.[2] The magnitude component clearly shows (rectangles of lighter shade in the gray-scale plot) that the functions do not overlap but just touch at the corners.

Figure 6 shows an intensity plot for a field that has a semiangle almost equal to π, in other words, almost the full circle. The spectral correlation function in Fig. 6(b) is again extracted from the intensity map. The magnitude component clearly shows almost complete overlap, except in an x–y cross pattern at the center. The actual position of the cross depends on the location of the gap in the radiation direction. The nonoverlapping region allows the phase to be recovered from the intensity information.

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Corresponding author C. J. R. Sheppard's e-mail address is colin@physics.usyd.edu.au.

*Also with Australian Key Center for Microscopy and Microanalysis, University of Sydney, Sydney, NSW 2006, Australia.

†Also with Canon Information Systems Research Australia, 1 Thomas Holt Drive, North Ryde, NSW 2113, Australia.

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