Wigner function for highly convergent three-dimensional wave fields

Colin J. R. Sheppard

Department of Physical Optics, School of Physics, University of Sydney, Sydney, New South Wales 2006, Australia, and Australian Key Centre for Microscopy and Microanalysis, University of Sydney, Sydney, New South Wales 2006, Australia

Kieran G. Larkin*

Department of Physical Optics, School of Physics, University of Sydney, Sydney, New South Wales 2006, Australia

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The angle-impact Wigner function for highly convergent three-dimensional scalar wave fields is derived directly by use of the three-dimensional generalized optical transfer function rather than from a six-dimensional Wigner function. The angle-impact Wigner function is a real four-dimensional function from which the intensity at any point in space is readily determined.

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There is strong current interest in the use of the Wigner function to describe optical propagation. Usually the Wigner function is developed by use of a paraxial approximation, but recently Wolf *et al.* presented a treatment based on a two-dimensional (2-D) nonparaxial scalar treatment.[1] Their approach was to derive an angle-impact marginal from the four-dimensional Wigner function. The aim of this Letter is twofold: to develop the angle-impact Wigner function for nonparaxial wave fields based on a full three-dimensional (3-D) scalar treatment and to derive it directly from the generalized optical transfer function (OTF),[2–4] rather than introducing a six-dimensional Wigner function. It is interesting to note that, although the concept of the generalized OTF was first proposed in the mid-1960s, its connection with the Wigner function does not seem to have been discussed previously.

The amplitude in the focal region of a convergent quasi-monochromatic field can be written in terms of the 3-D Fourier transform of a generalized pupil function $\prod(\mathbf{m})$ as[2]

$$U(\mathbf{r}) = (k/2\pi)^{3/2} \iiint_{-\infty}^{+\infty} \prod(\mathbf{m})\exp(ik\mathbf{m}\cdot\mathbf{r})\mathrm{d}^3\mathbf{m}\,, \quad (1)$$

where $k = 2\pi/\lambda$. As the field satisfies the scalar Helmholtz equation, the normalized spatial frequencies lie on the Ewald sphere, so that, in spherical coordinates m, θ, ϕ,

$$\prod(\mathbf{m}) = (2\pi/k)P(\theta,\phi)\delta(m-1)\,, \quad (2)$$

where m is the modulus of vector $\mathbf{m}$. The pupil $P(\theta,\phi)$ (or spectral function[1]) is the amplitude of the plane-wave component with direction cosines $\mathbf{m}$ coming from a point on the Ewald sphere with spherical coordinates θ, ϕ.

The intensity of the wave field is thus

$$I(\mathbf{r}) = (k/2\pi)^3 \iiiint\!\!\iint_{-\infty}^{+\infty} \prod(\mathbf{m}_1)\prod{}^*(\mathbf{m}_2) \times \exp[-ik(\mathbf{m}_2-\mathbf{m}_1)\cdot\mathbf{r}]\mathrm{d}^3\mathbf{m}_1\mathrm{d}^3\mathbf{m}_2\,, \quad (3)$$

where the spatial frequencies are illustrated in Fig. 1 and the generalized pupil function is taken as zero outside its passband. Setting

$$\mathbf{m} = \mathbf{m}_2 - \mathbf{m}_1\,,$$

$$\mathbf{p} = \frac{1}{2}(\mathbf{m}_2+\mathbf{m}_1), \quad p = \left(1-\frac{m^2}{4}\right)^{1/2}, \quad (4)$$

we have

$$I(\mathbf{r}) = \left(\frac{k}{2\pi}\right)^3 \iiiint\!\!\iint_{-\infty}^{+\infty} \prod\left(\mathbf{p}-\frac{\mathbf{m}}{2}\right)\prod{}^*\left(\mathbf{p}+\frac{\mathbf{m}}{2}\right) \times \exp(-ik\mathbf{m}\cdot\mathbf{r})\mathrm{d}^3\mathbf{m}\mathrm{d}^3\mathbf{p}\,. \quad (5)$$

Equation (5) can be written in terms of the 3-D generalized OTF:

$$I(\mathbf{r}) = \left(\frac{k}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} C(\mathbf{m})\exp(-ik\mathbf{m}\cdot\mathbf{r})\mathrm{d}^3\mathbf{m}\,, \quad (6)$$

where the 3-D generalized OTF is given by the autocorrelation of the generalized pupil,[5]

$$C(\mathbf{m}) = \iiint_{-\infty}^{\infty} \prod\left(\mathbf{p}-\frac{\mathbf{m}}{2}\right)\prod{}^*\left(\mathbf{p}+\frac{\mathbf{m}}{2}\right)\mathrm{d}^3\mathbf{p}\,, \quad (7)$$

i.e., as the overlap of two displaced Ewald spheres. The 3-D generalized OTF is Hermitian.

The intersection of the two spheres is a circle, so, in terms of the angle ψ representing the position on the circle,

$$C(\mathbf{m}) = \left(\frac{2\pi}{k}\right)^2 \int_{-\pi}^{\pi} \frac{P(\theta_1,\phi_1)P^*(\theta_2,\phi_2)}{|m|}\mathrm{d}\psi\,, \quad (8)$$

where $|m|$ in the denominator results from the effects of both the angle of intersection of the spherical surfaces, which gives a factor $1/|mp|$, and the radius $|p|$ of the circle of intersection. Substituting Eq. (8) back into Eq. (6), we obtain

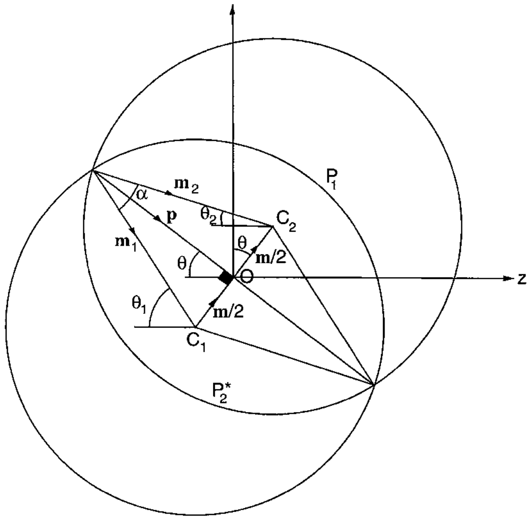

Fig. 1. Geometry of two intersecting generalized pupil functions, shown in the meridional plane containing vector **m**.

$$I(\mathbf{r}) = \frac{k}{2\pi}\iint_{-\infty}^{\infty}\int_{-\pi}^{\pi}\frac{P(\theta_1,\phi_1)P^*(\theta_2,\phi_2)}{|m|} \times \exp(-ik\mathbf{m}\cdot\mathbf{r})\mathrm{d}\psi\mathrm{d}^3\mathbf{m}$$

$$= \frac{k}{2\pi}\iint_{\Omega}\int_0^2\int_{-\pi}^{\pi}P(\theta_1,\phi_1)P^*(\theta_2,\phi_2) \times \exp(-ik\mathbf{m}\cdot\mathbf{r})\,|m|\mathrm{d}\psi\mathrm{d}m\mathrm{d}^2\Omega\,. \quad (9)$$

Putting

$$l = \frac{\mathbf{m}\cdot\mathbf{r}}{m} = \hat{\mathbf{m}}\cdot\mathbf{r}\,, \quad (10)$$

i.e., the projection of **r** in the direction of **m**, and introducing the spectral correlation function[6]

$$\gamma(\alpha,\theta,\phi,\psi) = P(\theta_1,\phi_1)P^*(\theta_2,\phi_2)\,, \quad (11)$$

where θ and α are shown in Fig. 1, so that

$$m = 2\sin(\alpha/2)\,, \quad (12)$$

we then have

$$I(\mathbf{r}) = \frac{k}{2\pi}\iint_{\Omega}\int_0^2\int_{-\pi}^{\pi}\gamma(\alpha,\theta,\phi,\psi)\exp(-ikml) \times |m|\mathrm{d}\psi\mathrm{d}\mathbf{m}\mathrm{d}^2\Omega\,. \quad (13)$$

Note that we have defined l as the scalar product of $\hat{\mathbf{m}}$ and **r**, but for the 2-D case Wolf *et al.*[1] consider l alternatively as the modulus of a vector product of $\hat{\mathbf{p}}$ and **r**. If we allow m, and therefore α, to take negative as well as positive values, then we integrate twice over the surface of the sphere. This is consistent with the quasi-polar coordinates of Alieva and Bastiaans[7] and Bracewell (Ref. 8, p. 521). So

$$I(\mathbf{r}) = \frac{k}{4\pi}\iint_{\Omega}\int_{-2}^{2}\int_{-\pi}^{\pi}\gamma(\alpha,\theta,\phi,\psi)\exp(-ikml) \times |m|\mathrm{d}\psi\mathrm{d}\mathbf{m}\mathrm{d}^2\Omega\,. \quad (14)$$

Performing the integral in m first, we then have

$$I(\mathbf{r}) = \frac{1}{2}\iint_{\Omega}\int_{-\pi}^{\pi}M(\theta,\phi,\hat{\mathbf{m}}\cdot\mathbf{r},\psi)\mathrm{d}\psi\mathrm{d}^2\Omega\,, \quad (15)$$

where the angle-impact marginal[1] of the Wigner function $M(\theta,\phi,l,\psi)$ (which we call the angle-impact Wigner function for brevity) is given by

$$M(\theta,\phi,l,\psi) = \frac{k}{2\pi}\int_{-2}^{2}\gamma(\alpha,\theta,\phi,\psi)\exp(-ikml)\,|m|\mathrm{d}m\,, \quad (16)$$

$$M(\theta,\phi,l,\psi) = \frac{k}{2\pi}\int_{-\pi}^{\pi}\gamma(\alpha,\theta,\phi,\psi)\exp\left(-2ikl\sin\frac{\alpha}{2}\right) \times |\sin\alpha|\mathrm{d}\alpha\,, \quad (17)$$

i.e., the angle-impact Wigner function as defined here is related, by use of Eq. (8), to the 3-D generalized OTF by, say,

$$\int_{-\pi}^{\pi}M(\theta,\phi,l,\psi)\mathrm{d}\psi = (k/2\pi)^3\int_{-2}^{2}C(\mathbf{m}) \times \exp(-ikml)m^2\mathrm{d}m = N(\theta,\phi,l)\,. \quad (18)$$

As $C(\mathbf{m})$ is Hermitian, the Wigner function is real. Essentially, this is the reason for integrating twice over the surface of the sphere, as integrating twice allows simplifying symmetry in all the integration variables.

The 2-D form of the angle-impact Wigner function is the basis of a direct method of phase retrieval.[9] The concept of the angle and impact parameters of a ray (in 3-D geometric optics) is exactly analogous to the familiar concepts of projection plane orientation and intercept in Radon-transform-based tomography. Although $C(\mathbf{m})$ can exhibit a singularity at the origin, as can be seen from Eq. (8), this singularity is removed in Eq. (18) by the factor m^2. Equation (18) is a Fourier transform if the generalized OTF is taken as zero outside its passband. Equation (18) can thus be inverted to give

$$C(\mathbf{m}) = \left(\frac{2\pi}{k}\right)^2\frac{1}{m^2}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}M(\theta,\phi,l,\psi)\exp(ikml)\mathrm{d}\psi\mathrm{d}l\,. \quad (19)$$

Equation (16) can also be inverted to give

$$\gamma(\alpha, \theta, \phi, \psi) = \frac{1}{2|\sin(\alpha/2)|} \int_{-\infty}^{\infty} M(\theta, \phi, l, \psi) \times \exp\left(2ikl \sin \frac{\alpha}{2}\right) \mathrm{d}l . \quad (20)$$

The wave-field amplitude can thus be recovered from $M(\theta, \phi, l, \psi)$ in a manner analogous to Eq. (36) of Wolf *et al.*[1]

The special case when $\alpha = \pi$ is interesting, as then the integral in ψ in Eq. (15) reduces to the evaluation at a single point. We then obtain

$$P(\pi - \theta, \pi - \phi)P^*(\theta, \phi) = \frac{1}{2} \int_{-\infty}^{\infty} M(\theta, \phi, l, \psi) \times \exp(2ikl)\mathrm{d}l \quad (21)$$

for any ψ.

So far we have not defined any coordinate system. However, often an optical system has an axis, which we can take in the z direction. The coordinate ψ can be defined relative to the plane containing $\mathbf{m}$ and $\mathbf{k}$. Then, also defining $\mathbf{m}$ in spherical coordinates as

$$\mathbf{m} = m(\cos\theta \cos\phi \mathbf{i} + \cos\theta \sin\phi \mathbf{j} + \sin\theta \mathbf{k}) , \quad (22)$$

where θ is measured from the equator (Fig. 1), we have

$$\begin{aligned} \mathbf{m}_{1,2} = & \left[\cos\frac{\alpha}{2}(\sin\psi \sin\phi - \sin\theta \cos\psi \cos\phi) \mp \sin\frac{\alpha}{2} \cos\theta \cos\phi\right]\mathbf{i} \\ & - \left[\cos\frac{\alpha}{2}(\sin\psi \cos\phi + \sin\theta \cos\psi \sin\phi) \pm \sin\frac{\alpha}{2}\cos\theta \sin\phi\right]\mathbf{j} \\ & + \left(\cos\frac{\alpha}{2}\cos\theta \cos\psi \mp \sin\frac{\alpha}{2}\sin\theta\right)\mathbf{k} , \end{aligned} \quad (23)$$

which are related to the spherical coordinates of the pupils by

$$\mathbf{m}_{1,2} = \sin\theta_{1,2} \cos\phi_{1,2}\mathbf{i} + \sin\theta_{1,2} \sin\phi_{1,2}\mathbf{j} + \cos\theta_{1,2}\mathbf{k} . \quad (24)$$

Also,

$$\sin\psi = \frac{\mathbf{p} \cdot \mathbf{m} \times \mathbf{k}}{m(1 - m^2/4)^{1/2}} = \mathbf{k} \cdot \mathbf{m}_1 \times \mathbf{m}_2 . \quad (25)$$

For the special case when the pupil is circularly symmetric, the pupils are a function of

$$\cos\theta_{1,2} = \cos\frac{\alpha}{2}\cos\theta \cos\psi \mp \sin\frac{\alpha}{2}\sin\theta \quad (26)$$

only. We also have

$$\mathrm{d}^2\Omega = \cos\theta \mathrm{d}\theta \mathrm{d}\phi . \quad (27)$$

As a simple example, we consider the case of a uniform spherically converging wave. Then we have, from Eq. (17), normalizing to unity for $l = 0$,

$$M(\theta, \phi, l, \psi) = 2\left(\frac{\sin 2kl}{2kl}\right) - \left(\frac{\sin kl}{kl}\right)^2 , \quad (28)$$

giving for the intensity in the focal region

$$I(r) = 2\pi\left(\frac{\sin kr}{kr}\right)^2 , \quad (29)$$

where $r = |\mathbf{r}|$.[10] The intensity can be seen to be spherically symmetric.

The Wigner function derived here for 3-D Helmholtz wave fields is analogous to the angle-impact Wigner function derived by Wolf *et al.*[1] for 2-D Helmholtz wave fields. However, in the present case it has been derived directly by use of the 3-D generalized OTF instead of by introduction of six-dimensional Wigner functions. A difference between our results for three dimensions and those of Wolf *et al.*[1] is the factor sin α in the integral of Eq. (16), which is absent from the 2-D treatment.

If the angular spectrum is confined to a hemisphere, the integral in Eq. (15) need only be evaluated over the hemisphere and the factor of 1/2 dropped:

$$I(\mathbf{r}) = \iint_{{m_x}^2 + {m_y}^2 < 2 - \pi} \int_{-\pi}^{\pi} \frac{M(\theta, \phi, \hat{\mathbf{m}} \cdot \mathbf{r}, \psi)}{m_z} \mathrm{d}\psi \mathrm{d}m_x \mathrm{d}m_y . \quad (30)$$

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*Present address, Canon Information Systems Research Australia, 1 Thomas Holt Drive, North Ryde, New South Wales 2113, Australia.

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