Wigner function and ambiguity function for non-paraxial wavefields

Colin JR Sheppard*[a,b], Kieran G Larkin§[a, c]
[a] School of Physics, University of Sydney; [b] Australian Key Centre for Microscopy and Microanalysis, University of Sydney; [c] Canon Information Systems Research Australia

ABSTRACT

The connection between the Wigner function and the generalized OTF, and between the ambiguity function and the generalized OTF is investigated for non-paraxial scalar wavefields. The treatment is based on two-dimensional (2-D) wavefields for simplicity, but can be extended to the three-dimensional case.

Keywords: diffraction, propagation, phase-space representation, Wigner function, ambiguity function, OTF, transforms

1. INTRODUCTION

In the paraxial regime, the Wigner function[1] and ambiguity function[2] are two phase-space representations that can be used to describe propagation of waves. For example Brenner, Lohmann and Ojeda-Castañeda[3] described how the ambiguity function can be used as a polar display of the defocused OTF. However, it is well known that, although these phase-space reprentations are useful in the paraxial regime, for highly-convergent fields their form changes upon propagation, an effect analogous to the introduction of aberrations. Wolf et al.[4] introduced a form of Wigner function for 2-D non-paraxial wavefields, called the angle-impact marginal. This has the properties that it is real and covariant under translation or rotation.

An alternative representation for wave propagation is based on the concepts of the generalized pupil function,[5] and the generalized OTF,[6-12] which are two- or three-dimensional functions for two- or three-dimensional wavefields, respectively. These generalized functions have been investigated in the paraxial regime, and for highly-convergent scalar and vector wavefields. We have found that the concept of the generalized OTF is useful in visualizing the derivation of the Wigner function.[13]

These different representations have also found use in the phase retrieval problem, where knowledge of the intensity in the focal region can be used to reconstruct the phase variations.[14-18]

2. DERIVATION OF THE ANGLE-IMPACT MARGINAL

We consider the two-dimensional problem of a scalar wave propagating in a plane. The amplitude in the focal region can be written as the Fourier transform of the generalized pupil:

$$U(\mathbf{r})=\frac{k}{2\pi}\int_{-\infty}^{+\infty}\!\!\int \Pi(\mathbf{m})\exp(ik\,\mathbf{m}\cdot\mathbf{r})\mathrm{d}^2\mathbf{m}, \tag{1}$$

*colin@physics.usyd.edu.au, fax +61 2 9351 7727, Physical Optics Department, School of Physics, University of Sydney, NSW 2006 Australia; §kieran@research.canon.com.au, fax +61 2 9805 2929, Canon Information Systems Research Australia, 1 Thomas Holt Drive, North Ryde, NSW 2113, Australia

where $k = 2\pi / \lambda$. The generalized pupil is zero except on the surface of the Ewald circle, so that

$$\Pi(\mathbf{m}) = \frac{2\pi}{k} P(\theta)\delta(m-1), \tag{2}$$

where m is the modulus of the vector $\mathbf{m}$. The intensity in the focal region is

$$I(\mathbf{r}) = \left(\frac{k}{2\pi}\right)^2 \iiiint_{-\infty}^{+\infty} \Pi(\mathbf{m}_1)\Pi^*(\mathbf{m}_2)\exp\left[-ik(\mathbf{m}_2 - \mathbf{m}_1)\cdot\mathbf{r}\right]\mathrm{d}^2\mathbf{m}_1\mathrm{d}^2\mathbf{m}_2. \tag{3}$$

Figure 1 shows the geometry in spatial frquency space.

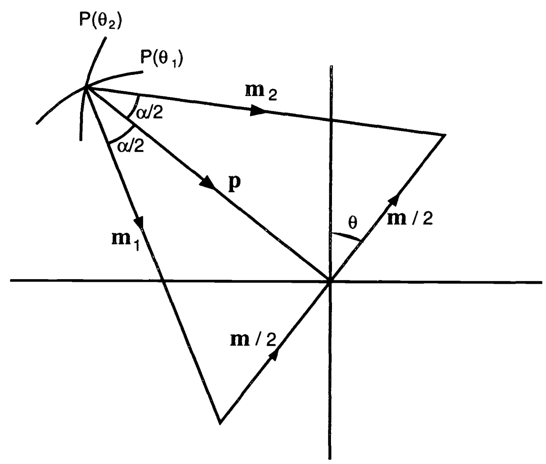

Fig.1 The geometry of two intersecting generalized pupil functions.

Putting

$$\mathbf{m} = \mathbf{m}_2 - \mathbf{m}_1$$

$$\mathbf{p} = \frac{1}{2}(\mathbf{m}_2 + \mathbf{m}_1), \quad p = \left(1 - \frac{m^2}{4}\right)^{1/2} \tag{4}$$

then the intensity is

$$I(\mathbf{r}) = \left(\frac{k}{2\pi}\right)^2 \iiiint_{-\infty}^{+\infty} \Pi\left(\mathbf{p} - \frac{\mathbf{m}}{2}\right)\Pi^*\left(\mathbf{p} + \frac{\mathbf{m}}{2}\right)\exp(-ik\mathbf{m}\cdot\mathbf{r})\mathrm{d}^2\mathbf{m}\mathrm{d}^2\mathbf{p}. \tag{5}$$

The intensity can also be written as the Fourier transform of the generalized OTF

$$I(\mathbf{r}) = \frac{k}{2\pi}\int\int_{-\infty}^{\infty} G(\mathbf{m})\exp(-ik\mathbf{m}\cdot\mathbf{r})\,\mathrm{d}^2\mathbf{m}, \tag{6}$$

so that

$$G(\mathbf{m}) = \frac{k}{2\pi}\int\int_{-\infty}^{\infty} \Pi\left(\mathbf{p}-\frac{\mathbf{m}}{2}\right)\Pi^*\left(\mathbf{p}+\frac{\mathbf{m}}{2}\right)\mathrm{d}^2\mathbf{p}, \tag{7}$$

or

$$C(\mathbf{m}) = \frac{2\pi}{k}\frac{P(\theta_1)P^*(\theta_2)}{|\sin\alpha|}, \tag{8}$$

where

$$m = 2\sin\frac{\alpha}{2}. \tag{9}$$

Thus

$$I(\mathbf{r}) = \int\int\int_{-\infty}^{\infty} \frac{P(\theta_1)P^*(\theta_2)}{|\sin\alpha|}\exp(-ik\mathbf{m}\cdot\mathbf{r})\ \mathrm{d}^2\mathbf{m}\ . \tag{10}$$

Putting

$$\ell = \frac{\mathbf{m}\cdot\mathbf{r}}{m} = \hat{\mathbf{m}}\cdot\mathbf{r}, \tag{11}$$

and defining the spectral correlation function[2] as

$$\gamma(\alpha,\theta) = P(\theta_1)P^*(\theta_2), \tag{12}$$

the intensity is then

$$I(\mathbf{r}) = \sqrt{\frac{2\pi}{k}}\int_{-\pi}^{\pi} M(\theta,\hat{\mathbf{m}}\cdot\mathbf{r})\,\mathrm{d}\theta, \tag{13}$$

where the angle-impact Wigner function is

$$M(\theta,\ell) = \sqrt{\frac{k}{2\pi}}\int_{-2}^{2}\gamma(\alpha,\theta)\frac{\exp(-ikm\ell)}{\sqrt{1-m^2/4}}\,\mathrm{d}m \tag{14}$$

$$= \sqrt{\frac{k}{2\pi}} \int_{-\pi}^{\pi} \gamma(\alpha,\theta) \exp\left(-2ik\ell \sin\frac{\alpha}{2}\right) \mathrm{d}\alpha, \qquad (15)$$

Inverting Eq.14 we then obtain

$$\gamma(\alpha,\theta) = \sqrt{\frac{k}{2\pi}} \cos\frac{\alpha}{2} \int_{-\infty}^{\infty} M(\theta,\ell) \exp\left(2ik\ell \sin\frac{\alpha}{2}\right) \mathrm{d}\ell, \qquad (16)$$

which allows the pupil function to be recovered from the Wigner function. In the full three-dimensional form of the derivation, some differences occur in the resulting equations.[19]

3. OTHER REPRESENTATIONS

The angle-impact Wigner function can be regarded as an intermediate step in the transformation between real and reciprocal space. The integrals in α and θ in Eqs.13, 15 can be alternatively performed in the opposite order. In this case a different intermediate function is obtained that reduces to the ambiguity function in the paraxial limit. In yet another approach, we start from the intensity rather than the spectral correlation function. Again different intermediate functions are generated that have analogies to the Wigner and ambiguity functions of the paraxial regime.

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