Joint Distribution Functions and the Generalized Optical transfer Function

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Abstract: The ambiguity function and the Wigner distribution function have both been applied in the optical area for many years. Later, the fractional Fourier transform has also been used. The connection between the ambiguity function and the defocused optical transfer function has also been described. Here we consider the connections with the generalized optical transfer function, first proposed in 1965, which is a two (three) dimensional optical transfer function for the two (three) dimensional case. The two dimensional form can be used as the basis for phase retrieval algorithms, but is also valid in the non-paraxial domain.

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INTRODUCTION

Joint distribution functions such as the Wigner distribution function and the ambiguity function are often used in optics. The ambiguity function was first applied to optical problems by Papoulis [1]. This paper discussed the propagation of the second moment width of a general light beam, and pointed out the property of the focal shift for optical systems of finite value of Fresnel number, well before this effect was generally recognized in the 1980s. Papoulis introduced the function □ defined as

$$\square(x,x') = \frac{1}{E} U(x + x'/2) U^{*}(x - x'/2), \tag{1}$$

where U is the amplitude and E is the energy of a beam. Papoulis did not give a name to the function □ The ambiguity function is given by its Fourier transform with respect to x. Brenner et al. have shown [2] that for paraxial optics there is a simple relationship between the ambiguity function and the defocused optical transfer function (OTF) when plotted in polar coordinates. The application of the Wigner distribution function in optics was first discussed in detail by Bastiaans [3, 4]. The Wigner distribution function is the Fourier transform of □ with respect to x'. Bastiaans discussed how this can be applied to the partially-coherent case when □ does not separate into a product, but here we will be mainly concerned with the fully coherent case. Knowledge of the

Wigner or ambiguity function can be used to recover the second order statistics of the wave field.

The fractional Fourier transform (FRFT), first described by Condon [5] has also been found useful in describing wave propagation. Mustard [6, 7] showed how the FRFT is related to the Wigner distribution. Lohmann [8] described how the FRFT corresponds to a rotation of the Wigner distribution, and presented optical schemes for generation of the FRFT. The Radon-Wigner transform was shown to be the squared modulus of the FRFT [9], that is, a projection of the Wigner function is the intensity of the FRFT. It was also shown how the Fourier transform of the intensity of the FRFT is a slice through the ambiguity function [10]. As the Wigner function and the ambiguity function are related by a two-dimensional (2D) Fourier transform relationship, this follows directly from the projection slice theorem.

An alternative way of describing wave fields is in terms of the generalized OTF (GOTF), which is the two-dimensional (2D) Fourier transform of the intensity for 2D wave fields (i.e. fields focused by cylindrical lens systems), and the 3D Fourier transform of the 3D intensity variation for the full 3D case [11, 12]. This concept follows from the concept of the generalized pupil function [13] as the OTF is the autocorrelation of the pupil function. The amplitude in the focal region is given by the 3D Fourier transform of the cap of a spherical shell, corresponding to part of the Ewald sphere in X-ray diffraction theory. It is interesting to note that the concept of the GOTF in optics is thus much older than application of ambiguity and Wigner functions, or the FRFT, in optics. McCutchen described how the amplitude along any line through the focal point is given by the Fourier transform of the projection of the pupil on to a parallel line through the centre of the sphere. The sphere degenerates into a paraboloid in the paraxial approximation. In the paraxial case there exist simple relationships between the joint distribution functions and the GOTF [14]. In fact the Wigner distribution function and the ambiguity function can both be considered as intermediate steps between the intensity point spread function and the GOTF. Frieden [12] showed that in the paraxial regime the GOTF is given by a straight-line integral across the product of two displaced pupils, one complex conjugated. A useful feature of the GOTF is that is readily extended to the non-paraxial case [15], whereas the large angle generalization of the usual definitions of the joint distribution functions is not straightforward. The straight-line integral approach to determining the GOTF is no longer valid in the non-paraxial regime

For a paraxial, convergent quasi-monochromatic wavefield, the defocused pupil function is

$$P(m,n;W_{20}) = P_0(m,n)\exp\left[ikW_{20}(m^2+n^2)\right], \tag{2}$$

where $P_0(m,n)$ is the in-focus pupil function, m, n are normalized spatial frequencies so that the cut-off is $m^2+n^2=1$, W_{20} is the defocus coefficient, and $k=2\pi/\lambda$. We now introduce the function Γ defined in spatial frequency space, which we call the spectral correlation function:

$$\Gamma(m,n;m',n') = \frac{1}{E} P_0(m'+m_0/2, n'+n_0/2) \\ \times P_0^*(m'-m_0/2, n'-n_0/2), \tag{3}$$

The defocused OTF is thus

$$C(m,n;W_{20}) = \iint \Gamma(m,n;m',n') \\ \times \exp\left[2ikW_{20}(mm'+nn')\right] dm'\,dn' \tag{4}$$

and the ambiguity function is

$$A(m,n;x,y) = \iint \Gamma(m,n;m',n') \\ \times \exp\left[i2\pi(m'x+n'y)\right] dm'\,dn'. \tag{5}$$

Thus

$$C(m,n,W_{20}) = A\left(m,n;2W_{20}m/\lambda, 2W_{20}n/\lambda\right). \tag{6}$$

We see that knowledge of the complete defocused OTF determines only a 3D section through the 4D ambiguity function. Thus measurement of the 3D wave-field [16] does not provide sufficient information to recover the full details of the wavefield in the general partially-coherent case, as was pointed out by Hazak [17] and Gori et al. [18]. For the coherent case, on the other hand, the phase can be reconstructed from 3D intensity information using the concept of the transport equation of intensity [19-21].

The 3D GOTF is

$$G(m,n,s) = 2\lambda \iint \Gamma(m,n;m',n') \\ \times \delta\left[s - (mm'+nn')\right] dm'\,dn', \tag{7}$$

so that for the paraxial case it is given by a line integral on a straight line across the region of overlap of two displaced pupils, one complex conjugated, as originally pointed out by Frieden [12]. For the 2D case this reduces to

$$G(m,s) = \frac{2\pi}{|m|}\gamma(m,m');\, m' = s/m\,. \tag{8}$$

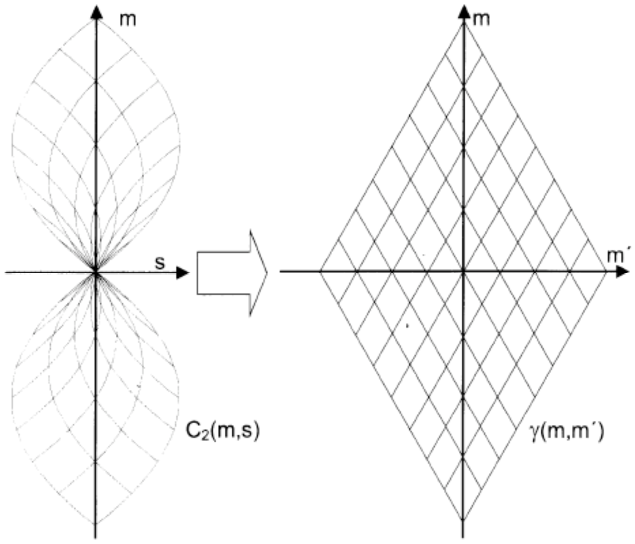

FIGURE 1. The coordinate transformation between the 2D generalized OTF in m, s and the spectral correlation function $\gamma(m,m')$ for the paraxial regime. Reproduced from [14].

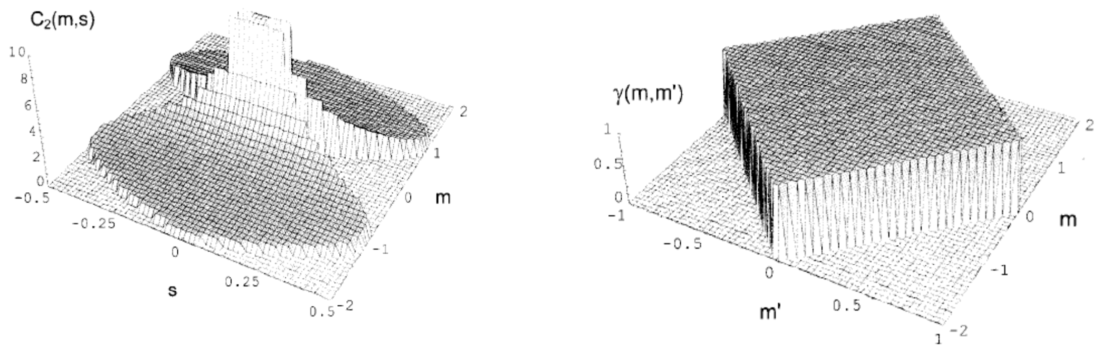

FIGURE 2. The 2D generalized OTF and the spectral correlation function for an unaberrated slit pupil in the paraxial regime. Reproduced from [14].

For the 2D nonparaxial case the amplitude in the focal region can be written as an angular spectrum of plane waves

$$U(\mathbf{r}) = \frac{k}{2\pi}\iint \mathcal{P}(\mathbf{m})\exp(ik\mathbf{m}\cdot\mathbf{r})\,d^2\mathbf{m}. \tag{9}$$

Because the wave-field satisfies the Helmholtz equation, the generalized pupil is zero except on the surface of the Ewald circle (the 2D form of the Ewald sphere), so that

$$\mathcal{P}(\mathbf{m}) = P(\theta)\delta(|\mathbf{m}| - 1)\,. \tag{10}$$

Putting

$$\mathbf{m} = \mathbf{m}_1 - \mathbf{m}_2; \quad \mathbf{p} = (\mathbf{m}_1 + \mathbf{m}_2)/2, \tag{11}$$

the intensity is

$$I(\mathbf{r}) = \left(\frac{k}{2\pi}\right)^2 \iiiint P\left(\mathbf{p} + \frac{\mathbf{m}}{2}\right) P^*\left(\mathbf{p} - \frac{\mathbf{m}}{2}\right) \exp(ik\mathbf{m}\cdot\mathbf{r})\, d^2\mathbf{m}\, d^2\mathbf{p} \tag{12}$$

so that the GOTF is

$$G(\mathbf{m}) = \frac{k}{2\pi} \iint P\left(\mathbf{p} + \frac{\mathbf{m}}{2}\right) P^*\left(\mathbf{p} - \frac{\mathbf{m}}{2}\right) d^2\mathbf{p}, \tag{13}$$

i.e. the autocorrelation of the generalized pupil, as shown in Fig.3. Thus

$$G(\mathbf{m}) = \frac{2\pi}{k} \frac{\Gamma(\theta,\psi)}{|\sin\theta|} = \frac{2\pi}{k} \frac{\Gamma(\theta,\psi)}{K\left(1 - K^2/4\right)^{1/2}}; K = |\mathbf{m}| = 2\sin\frac{\theta}{2}; \frac{s}{m} = \tan\psi, \tag{14}$$

where the spectral correlation function is defined for the nonparaxial case

$$\Gamma(\theta,\psi) = P(\alpha_1)P^*(\alpha_2) = P(\psi + \theta/2)P^*(\psi - \theta/2); \quad \theta = \alpha_1 - \alpha_2. \tag{15}$$

Note that the arguments are angles rather than the sines in the Papoulis treatment. The phase of the pupil function can thus be recovered from 2D intensity measurements even in the non-paraxial case [22].

In general the two circles intersect in two points, but if we restrict our attention to the case of forward propagating waves only, the arcs intersect in only one point. The full case of backward and forward propagating waves is discussed elsewhere. [23] For forward propagating waves, Eq. 14 can be inverted, to give

$$\Gamma(\theta,\psi) = \frac{k}{2\pi}|\sin\theta|\, G\left(2\sin\frac{\theta}{2}\cos\psi, 2\sin\frac{\theta}{2}\sin\psi\right). \tag{16}$$

The intensity can be written

$$\left|U(x,z)\right|^2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Gamma(\theta,\alpha)\exp\left[2ik\sin\frac{\alpha}{2}(x\cos\theta + z\sin\theta)\right]\mathrm{d}\theta\,\mathrm{d}\alpha. \tag{17}$$

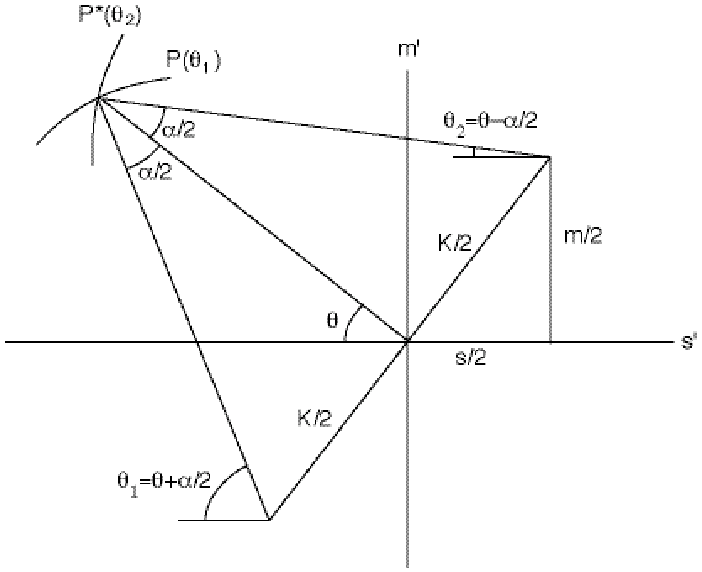

FIGURE 3. The GOTF given by the area of overlap of two displaced arcs of circles. Reproduced from [23].

Eq.17 is a double integral, which can be performed by evaluating the integral in α first to give

$$M(\theta,\ell) = \left(\frac{k}{2\pi}\right)^{1/2}\int_{-\pi}^{\pi}\Gamma(\theta,\alpha)\exp\left[2ik\ell\sin\frac{\alpha}{2}\right]\mathrm{d}\alpha, \tag{18}$$

so that

$$\left|U(x,z)\right|^2 = \left(\frac{2\pi}{k}\right)^{1/2}\int M(\theta, x\cos\theta + z\sin\theta)\,\mathrm{d}\theta. \tag{19}$$

Eq.18 defines Wolf *et al.*'s angle-impact marginal [24], which we call the Wigner function for simplicity. In the paraxial limit it reduces to the ordinary Wigner function $W(m', x)$.

Inverting Eq.18

$$\Theta(\phi,\theta) = \left(\frac{k}{2\pi}\right)^{1/2} \cos\frac{\theta}{2} \int_{-\infty}^{\infty} M(\phi,\ell)\exp\left(-2ik\ell\sin\frac{\theta}{2}\right)\mathrm{d}\ell . \tag{20}$$

We note that for $\ell = 0$, Eq.20 reduces to the circular autocorrelation of the pupil function. Putting $\theta = 0$

$$\left|P(\phi)\right|^2 = \left(\frac{k}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} M(\phi,\ell)\,\mathrm{d}\ell . \tag{21}$$

Alternatively, putting θ_1 or θ_2 constant in Eq.20 allows the angular spectrum to be recovered from the Wigner function to within a constant phase factor. We also have

$$\int \left|P(\theta)\right|^2 \mathrm{d}\theta = \frac{k}{2\pi}\int \left|U(x,z)\right|^2 \mathrm{d}\ell = \left(\frac{k}{2\pi}\right)^{1/2} \iint M(\theta,\ell)\,\mathrm{d}\ell\,\mathrm{d}\theta = E \tag{22}$$

is the total energy, a constant invariant under translation or rotation.

For the case of forward propagating waves only, using Eq.16 we can obtain an explicit Fourier relationship between the generalized OTF and the Wigner function:

$$G(m,s) = \left(\frac{2\pi}{k}\right)^{1/2} \frac{1}{(m^2+s^2)^{1/2}} \int_{-\infty}^{\infty} M\left(\arctan\left(\frac{s}{m}\right),\ell\right)\exp(-ik\ell K)\,\mathrm{d}\ell . \tag{23}$$

For a wave-field consisting of forward and backward propagating components, this is not in general possible as the two circles in the GOTF construction intersect in two points. The extension to the 3D case has been discussed elsewhere [25]. Now the two spheres in the GOTF construction intersect in a circle. Possible extensions for the ambiguity function have also been discussed [26].

DISCUSSION

Simple relationships exist between the Wigner function, the ambiguity function, the spectral correlation function, and the GOTF in the paraxial regime. In the non-paraxial regime, there is still a simple relationship between spectral correlation function and the GOTF. The angle-impact Wigner function is also still related to the spectral correlation function by a Fourier transformation. Using the spectral correlation function the phase can be retrieved directly from the intensity in the 2D case even for the non-paraxial regime (as long as the modulus of the angular spectrum is zero at some angle). For the 3D case, the phase can be retrieved from 3D intensity measurements, but the full mutual coherence function cannot be retrieved.

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