The Riesz transform in image processing

Applications and extensions:

A whistle-stop tour…

Kieran G. Larkin

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A Different Perspective?

Visual Pun by Ingram Pinn
Main Sections

– History/background in signal/image processing

– Riesz transform multiplier & its complex formulation
  
  • Demodulation of 2-D fringe patterns

– Orientation estimation and the energy operator in 1D, 2D, nD

– The Mellin transform extension (image watermarking)
Hilbert transform in Communication

• Modulated signals must be demodulated
  – Radio (AM and FM)
  – TV
  – Modems (Modulator/Demodulator)

\[ s(t) = a(t) \cos[\psi(t)] \]

• Direct demodulation uses Hilbert transform:

\[ H\{s(t)\} \approx -a(t) \sin[\psi(t)] \]


Obtain exact bounds on accuracy from Fourier transforms of signal
Can demodulation be applied to two-dimensional AM-FM patterns?
Ubiquitous Fringe Patterns
Fingerprint as fringe pattern
Fingerprint = Fringe Pattern

- Synthesis is trivial
- Fingerprint = 4 elemental images
- Each elemental image is highly redundant (compressible):

$$f(x, y) = a(x, y) + b(x, y) \cos[\psi_{\text{smooth}}(x, y) + \sum \theta_n]$$

Offset  Amplitude  Smooth Phase  Spiral phases
Fringe Pattern Disadvantage

- 2-D analysis is *difficult* (until 2000 anyway)

\[ f(x, y) = a(x, y) + b(x, y)\cos[\pm \psi (x, y)] \]

<table>
<thead>
<tr>
<th>Offset</th>
<th>Amplitude</th>
<th>Total Phase</th>
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<td>Sign Ambiguity</td>
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Natural Demodulation to the rescue

- Direct method for demodulating 2-D fringe patterns to find phase  

\[ f_0(x, y) = b(x, y) \cos[\psi(x, y)] \]

Spiral phase operator (Riesz multiplier)

\[ \mathbf{F}^{-1}\{e^{i\phi(u,y)} \mathbf{F}\{f_0(x,y)\}\} = -ie^{i\beta(x,y)}b(x,y)\sin[\psi(x,y)] \]
Phase and Amplitude found in one operation

\[ b(x, y)e^{i\psi(x, y)} = f_0(x, y) - \mathbf{F}^{-1}\left\{ e^{-i\beta(x, y)}e^{i\phi(u, v)}\mathbf{F}\{f_0(x, y)\}\right\} \]

Real

Imaginary

Space-Frequency operator/kernel?

\[ e^{-i\beta(x, y)}e^{i\phi(u, v)} \]

Kernel

\[ b(x, y)e^{i\psi(x, y)} = f_0(x, y) - \mathbf{F}^{-1}\{K(u, v, x, y).\mathbf{F}\{f_0(x, y)\}\} \]
Approximation Error

\[ f_0(x, y) = b(x, y) \cos[\psi(x, y)] \]

Riesz multiplier

\[ \mathbf{F}^{-1}\left\{ e^{i\phi(u, y)} \mathbf{F}\{ f_0(x, y) \} \right\} \approx -ie^{i\beta(x, y)} b(x, y) \sin[\psi(x, y)] \]

Asymptotic method of stationary phase gives estimation error

\[
\psi_{\varepsilon}(x, y) = \frac{\psi_{1,0}^2 \psi_{0,2}^2 + \psi_{0,1}^2 \psi_{20}^2 - 2\psi_{1,0} \psi_{0,1} \psi_{1,1}}{\left(\psi_{1,0}^2 + \psi_{0,1}^2\right)^{3/2}} \cdot \frac{1}{\left(\psi_{1,0}^2 + \psi_{0,1}^2\right)^{1/2}}
\]

phase curvature

|phase gradient|
How do we find Direction Map

\[ e^{i\beta (x,y)} \]
Fingerprint Phase Image Generation

Real | Imaginary | Modulus | Phase

$b \cdot \cos \psi$ | $i \cdot b \cdot \sin \psi$ | $|b|$ | $\psi$
Helmholtz Decomposition

- Want to split into smooth phase + singularities (Helmholtz)
- Find singularities by summing phase gradient around every pixel
- Spiral phase sums to +1 or -1
- Remove nearby opposite pairs (dipoles)
Construct Singularity Image

- Details omitted...see Ghiglia and Pritt for outline.
Spiral phase map with charges superimposed

positive
negative
Subtracting Singularities

• Singularities removed by subtracting spirals from phase image

• Decomposition is unique by Helmholtz

\[ \text{Total Phase} \quad \text{minus} \quad \text{Spiral Phase} \quad \text{equals} \quad \text{Wrapped Smooth Phase} \]
Unwrapping the smooth phase

• Without singularities the phase is *trivially* unwrapped
Compression Synthesis

\[ a(x,y) \quad b(x,y) \quad \psi_{smooth} \quad \psi_{spiral} \]

- Demodulate
- Recombine

262 kbytes → 4 kbytes

0.7 kbytes → 1.2 kbytes → 2 kbytes → 0.6 kbytes
Another example 239x compression

262144 bytes

1095 bytes
How do we find Direction Map $\beta(x,y)$?

- Robustly find orientation $2\beta$ (modulo $2\pi$) using uniform energy operator (Larkin, Opt. Exp. 2005, US Pat 7043082B2)
  $\exp[2i\beta(x,y)]$

- Unwrap orientation map to find direction $\beta$ (modulo $2\pi$).

$$\sqrt{\exp[2i\beta]} = \pm \exp[i\beta]$$

- Direction discontinuities. Sherlock and Monro (1993) showed how to unwrap (locally resolve sign $\pm$)
Riesz transforms to the rescue...

- Also unite two (previously disparate, but parallel) theories in signal processing:
  - Phase congruency (or local energy)
  - Energy operator

- Two ways to measure the energy of a (1-D) signal, both have problems with 2-D extensions
Local energy

• *Peter Kovesi, UWA, c.1991*

• **Observation:**
  
  at edges all the Fourier components are in phase

• **Equivalence**

  “*local*” energy defined by analytic signal is a good measure of edge significance

\[
E = \left[ s(x) \right]^2 + \left[ \mathcal{H}\{s(x)\} \right]^2
\]

Ironically nonlocal due to Hilbert kernel

Energy operator

- **Teager 1989, Kaiser 1990**
- **Vocal energy**
  
  *Simple harmonic oscillator model: potential + kinetic energy*

\[
\begin{align*}
s(t) &= b(t) \cos(\omega t) \\
s'(t) &\approx -b(t) \sin(\omega t) \\
s''(t) &\approx -b(t) \cos(\omega t)
\end{align*}
\]

\[
s'(t) s'(t) - s(t) s''(t) \approx \omega^2 b^2(t)
\]

- **Total energy**

2-D Energy Measures

• Failure of attempts: directionality/non-isotropy
• IDEA: use Riesz isotropy “trick”
• Constrain to complexified form: complex gradient

\[
D = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \leftrightarrow \text{FT} \rightarrow 2\pi i(u + iv) = 2\pi iqe^{i\phi}
\]

• Then simple extension has all required properties

\[
\Psi \{s\} \equiv \left[Ds\right]^2 - sD^2s
\]
2-D Energy Operator

- \[ \Psi \{ s \} \equiv [Ds]^2 - sD^2s \]

- Directional pattern

\[ s(x, y) = b(x, y) \cos[\psi(x, y)] \]

\[ \psi(x, y) = \omega(x \cos \beta + y \sin \beta) \]

\[ \Psi \{ s \} \approx b^2(x, y) \omega^2(x, y) e^{2i\beta(x, y)} \]

- Amplitude
- Frequency
- Orientation
Riesz transform formulation of EnOp

Energy operator

\[ \Psi \{s\} \equiv [Ds]^2 - sD^2s \]

Modified for Riesz operator

\[ \Psi_R \{s\} \equiv [D_Rs]^2 - sD^2_Rs \]

Nice result:

\[ \Psi_R \{s\} \approx b^2(x, y).e^{2i\beta(x, y)} \]

amplitude orientation
Energy operators summarised

- 2-D operators give amplitude/frequency/orientation

- Quadratic combination of Riesz 1 & 2

- Generalise order of convolution kernel (Riesz-like)

- Generalise Fourier multipliers

$$q^\alpha e^{i\phi}$$

$$\left(q^\alpha\right)^2 e^{2i\phi}$$

Gelfand and Shilov Homogeneous Functions and Fourier transforms
Noisy orientation estimation

gradient

EnOp

Riesz EnOp
EnOp Positivity (1-D)

Energy operator

\[
\mathbb{E}\{ f \} = (f')^2 - ff''
\]

Positivity (log concavity/convexity/subharmonicity)

\[
\mathbb{E}\{ f \} \equiv - \frac{f^2}{2} D^2 \log(f^2) \geq 0
\]

Attenuation form of signal

\[
f(t) = e^{\rho(t)}(t)\cos\phi(t)
\]
EnOp Positivity

Each angle defines a line

\[ E \{ f \} = \exp[2 \rho \phi] \left[ (\phi')^2 - \rho'' \cos^2 \phi + \phi'' \sin \phi \cos \phi \right] \geq 0 \]

Define normalised Laplacians

\[
\frac{\rho''}{(\phi')^2} = \alpha, \quad \frac{\phi''}{(\phi')^2} = \beta
\]
EnOp Positivity

\[ \beta = \frac{\phi''}{(\phi')^2} \]

\[ \alpha = \frac{\rho''}{(\phi')^2} \]
Imaginary Powers of Laplacian

• Homogeneous functions (see Gelfand & Shilov, vol 1)

• Fourier (spectral) Multiplier $\rightarrow$ self-Fourier

$$r^{p+ i\alpha} e^{ik\theta} \xrightarrow{FT} \gamma q^{2- p- i\alpha} e^{ik\phi}$$

• Mellin basis, LRHF, hyperbolic chirp, homogeneous function
  
  – Perfect correlation and orthogonality properties, Low visibility, RST invariant.

• 1-D form

$$|x|^{p+ i\alpha} \text{sgn}(x) \xrightarrow{FT} i\gamma^+ |u|^{1- p- i\alpha} \text{sgn}(u)$$

$$|x|^{p+ i\alpha} \xrightarrow{FT} \gamma^- |u|^{1- p- i\alpha}$$
Some real LRHFs

Pure Radial

Pure Circular

Mixed radial and circular

Small R

Large R
Diversion: applications watermarks

Fourier-Mellin basis functions are scale and rotation invariant

Other applications: fast and accurate alignment (correlation detection)

Kieran Larkin   10 March 2009
Equi-Angular Spirals!
Diversion: Riesz transform kernels

Scale and rotation invar/ariant
2D Basis Functions: cosine/sine 4D

Log-Polar coordinate transform yields Fourier-Mellin basis functions
Lots more I can’t fit in

• Complex derivatives (pseudodifferential operators)
• Helmholtz decomposition and reconstruction
• Monogenic functions
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