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On the positivity of an energy operator:

the interplay of amplitude and phase modulation in the Teager-Kaiser energy operator

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Motivated purely by curiosity we consider conditions for positivity of the Teager-Kaiser energy operator (TKEO) in one dimension. The TKEO has numerous applications in signal analysis, especially demodulation theory. Positivity underpins the definition of an ideal energy operator – a negative energy signal is essentially meaningless - but the TKEO only approximates the ideal. Formulating the problem logarithmically in terms of attenuation and phase, rather than the more conventional amplitude and phase we derive a family of normalised curvature constraints. The interplay of attenuation and phase then defines a deceptively simple 2-D parabolic domain outside of which positivity generally fails.

Introduction

In 1989 a simple operator, analogous to the total energy of a simple harmonic oscillator, was introduced to signal processing by Teager and Teager [1]. The main properties were soon explored and described by Kaiser [2]. By 1994 the main properties of the energy operator had been exposed and the conditions required for positivity determined [3]. A diverse range of applications for the Teager-Kaiser energy operator have been proposed, such as the analysis of optical interferograms [4], and the demodulation of wideband radio signals [5]. Recently we have become aware of an alternative way to view the energy operator [6], and accordingly, the positivity constraints. It is now possible to illustrate the separate constraints upon amplitude modulation and phase modulation as a subtle interplay.

Attenuation and the Logarithmic Formulation

Our approach is to develop the initial analysis of Bovik and Maragos [3] and show that their log-concavity signal condition implies an interrelation between amplitude and phase. We limit ourselves to continuous signals and the corresponding continuous energy operator. We can represent (zero-mean) amplitude and phase modulated signal (sometimes known as AM-FM):

$$f(t) = b(t).\cos\phi(t) \quad (1)$$

Log-concavity implies that an attenuation representation may be advantageous, and we drop the explicit temporal variation of Eq (1):

$$\log b = \rho \Rightarrow f = e^{\rho}\cos\phi \quad (2)$$

We can avoid negative attenuation, or π radian phase flipping, by squaring the signal:

$$f^2 = e^{2\rho}\cos^2\phi \geq 0 \quad (3)$$

We now define the energy operator and its real logarithmic formulation:

$$\mathrm{E}\{f\} = (f')^2 - ff'' = -\frac{1}{2} f^2 \frac{d^2}{dt^2}\left[\log\left(f^2\right)\right] \quad (4)$$

The positivity condition for an energy operator is simply E≥0. Utilizing the simplification of the attenuation form (in Eq.2), as originally described in the author's previous work [6]:

$$\log\left(f^2\right) = 2\rho + \log\left(\cos^2\phi\right) \quad (5)$$

Hence:

$$\mathrm{E}\{f\} = e^{2\rho}\left((\phi')^2 - \rho'' \cos^2\phi + \phi'' \sin\phi \cos\phi\right) \geq 0 \quad (6)$$

The positivity of the first factor is guaranteed, and the second factor has four remaining variables $\{\phi, \phi', \phi'', \rho''\}$. Interestingly the positivity depends neither on the either attenuation ρ nor its derivative ρ'. For some applications we may wish to convert back to the amplitude formulation, but some essential simplicity is lost in the following relations:

$$2\rho = \log(b^2), \Rightarrow \rho' = b'/b, \Rightarrow \rho'' = \frac{bb'' - (b')^2}{b^2} \quad (7)$$

Positivity and an Idealized Energy Operator

Returning now to the positivity of Eq.(6) everywhere except at zero-crossings of the amplitude, we must have the condition:

$$(\phi')^2 - \rho'' \cos^2\phi + \phi'' \sin\phi \cos\phi \geq 0 \quad (8)$$

Consider an idealized energy operator that we define to have the following property:

$$\mathrm{E}_{ideal}\{b\cos\phi\} = b^2(\varphi')^2 = (b\varphi')^2 \quad (6a)$$

The idealization guarantees positivity because the output is the square of a real function. The Teager-Kaiser realization actually gives the result in Eq(6), which we can rewrite:

$$\mathrm{E}\{f\} = \mathrm{E}_{ideal}\{f\}\left(1 - \frac{\rho'' \cos^2\varphi}{(\varphi')^2} + \frac{\varphi'' \sin\varphi \cos\varphi}{(\varphi')^2}\right). \quad (6b)$$

Returning to Eq(8) we observe that unless the first term (the ubiquitous square of the local frequency) is non-zero the positivity condition cannot be true for all values of the phase $-\pi \leq \phi < \pi$, independently of the two second derivatives $\{\phi'', \rho''\}$ (i.e. unless $\phi'' = 0, \Rightarrow \rho'' \leq 0$).

We are interested in the more general non-zero frequency case. The form of Eq(6b) suggests that we can reduce the number of variables by defining normalized curvatures α, β, analogous to normalized chirp rates:

$$(\varphi')^2 \neq 0, \quad \frac{\rho''}{(\varphi')^2} = \alpha, \quad \frac{\varphi''}{(\varphi')^2} = \beta \quad (9)$$

The result is a polar-like inequality in terms of the phase ϕ

$$\frac{1}{\cos\phi} \geq \alpha\cos\phi - \beta\sin\phi = (\mathbf{i}\alpha + \mathbf{j}\beta).(\mathbf{i}\cos\phi - \mathbf{j}\sin\phi). \tag{10}$$

Note the phase symmetry:

$$\frac{1}{\cos[\pm\phi]} \geq \alpha\cos\phi \mp \beta\sin\phi = (\mathbf{i}\alpha + \mathbf{j}\beta).(\mathbf{i}\cos\phi \mp \mathbf{j}\sin\phi). \tag{10A}$$

Eq(10) is the vector equation of a sequence or family of lines. Values of phase ($-\pi \leq \phi < \pi$) parameterize each line with a normal orientation ϕ. We can plot symmetric pairs of lines as shown in Fig.1.

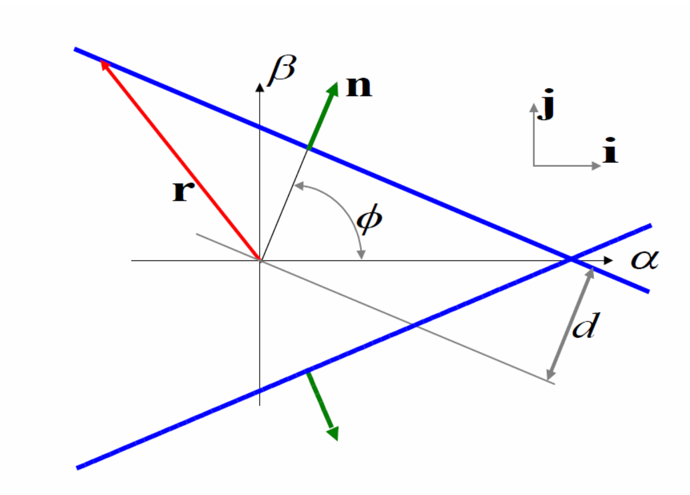

Fig 1. Vector equation of a line from perpendicular vector and distance.

Remember that the vector equation of a line can be written:

$$\mathbf{r}.\mathbf{n} = d \tag{11}$$

Here the position vector on the line is **r,** the perpendicular distance of the line from the origin is d , and the line perpendicular unit vector is **n**. From Eq(10) and Eq(11) we explicitly write the vector relations:

$$\left.\begin{aligned} \mathbf{n} &= \mathbf{i}\cos\phi + \mathbf{j}\sin\phi \\ \mathbf{r} &= \mathbf{i}\alpha + \mathbf{j}\beta \\ d &= 1/\cos\phi \end{aligned}\right\} \tag{12}$$

Although each line equation is rather simple, it is perhaps not immediately obvious what the overall sequence of lines actually means. Using Mathematica to plot out the family of lines for different values of phase φ we find that there is a tangent curve (or caustic) that uniquely defines regions of positivity and negativity of the energy operator. The red curve in Fig. 2 is the limit curve, or envelope, for all values of phase. It is reminiscent of a conic section.

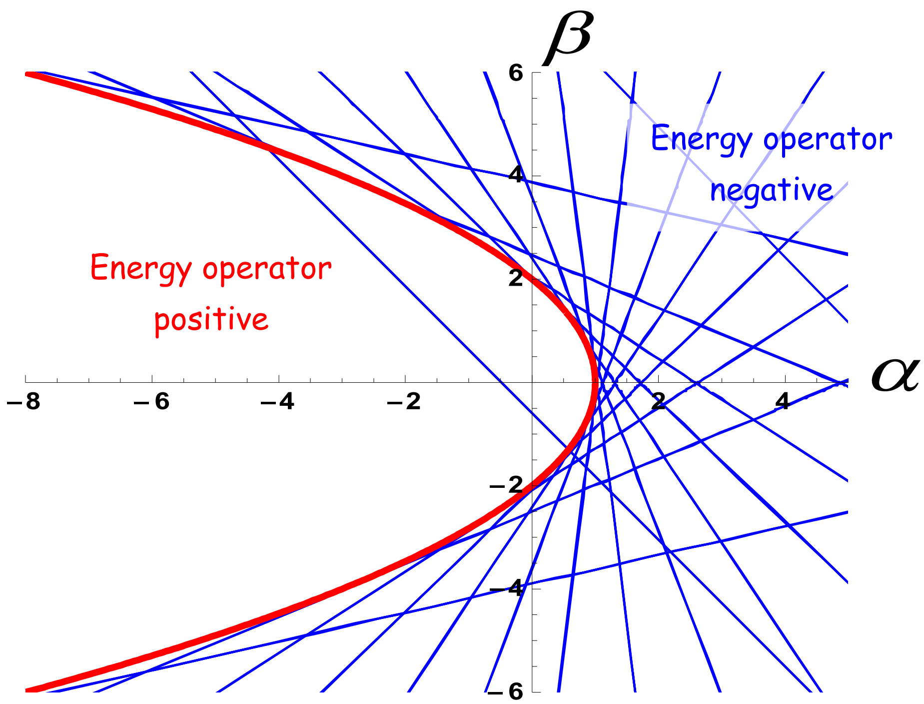

Fig 2. Region of positivity for attenuation and phase second derivatives.

It can be shown by a simple, but rather lengthy derivation (which we omit for brevity) that the positivity region is bounded by a parabola of the following form:

$$\alpha < 1 - \left(\frac{\beta}{2}\right)^2 \iff \beta < 2\sqrt{1-\alpha} . \tag{13}$$

Interpretation of Amplitude and Phase Modulation Constraints

Before going further, we note that three parameters have no bearing on the parabolic positivity boundary curve, namely ρ, ρ', φ. The first corresponds to a gain factor, the second to a time exponential factor, and the third reflects our original requirement for phase offset invariance. Below we consider a few specific regions of the limit curve where the energy operator is forced to zero. Note that Bovik and Maragos discussed positivity solutions containing simple exponentials (of order one) but omitted exponentials of order two. Their more general solution has a “sufficiently smooth” signal amplitude defined by the signal's logarithmic concavity.

Axes

The simple limits for pure amplitude or pure phase modulation occur along the x and y axes respectively of Fig.2:

$$\begin{cases} \beta = 0, \Rightarrow -\infty < \alpha < 1 \\ \alpha = 0, \Rightarrow -2 < \beta < 2 \end{cases} \tag{14}$$

Clearly there is an interplay or trade-off between modulation types. As the relative attenuation curvature $\alpha = \rho''/(\phi')^2$ becomes more negative $\alpha \to -\infty$, the allowable (relative) phase curvature range increases without limit $\beta \to \pm\infty$. There is the question of how long any of these extreme second derivatives can sustain. We can measure this in terms of nominal oscillation period $\tau = 2\pi/\phi'$

Condition A, Y-Axis: constant or exponential amplitude

Consider a local expansion, up to second derivatives:

$$\alpha = 0, |\beta| \le 2, \ \Rightarrow \rho_2 = 0, \ \Rightarrow f(t) = \exp(\rho_0 + \rho_1 t)\cos(\varphi_0 + \varphi_1 t \pm 2\varphi_1^2 t^2) \tag{15}$$

$$\cos(\varphi_0 + \varphi_1 t \pm 2\varphi_1^2 t^2) = \cos(\varphi_0 + \varphi_1 t[1 \pm 2\varphi_1 t]) \tag{15a}$$

which shows the maximum chirp rate for a simple exponential envelope.

Condition B, decaying Gaussian amplitude

Local expansion:

$$\alpha \to -\infty, |\beta| < 2\sqrt{1-\alpha}, \ \Rightarrow f(t) = \exp(\rho_0 + \rho_1 t + \alpha\varphi_1^2 t^2)\cos(\varphi_0 + \varphi_1 t \pm 2\sqrt{1-\alpha}\varphi_1^2 t^2) \tag{16}$$

which shows the maximum chirp rate for a rapidly attenuated (Gaussian) envelope increases approximately as the square root of the amplitude attenuation Laplacian. Note, however that the Gaussian width (sigma) is much less than the nominal oscillation period τ.

Condition C, constant frequency, exponentially growing amplitude

Local expansion:

$$\beta = 0, \ \alpha = 1 \Rightarrow f(t) = \exp(\rho_0 + \rho_1 t + \rho_1^2 t^2)\cos(\varphi_0 + \varphi_1 t) \tag{17}$$

which is a fixed frequency signal with a reciprocal Gaussian (quadratic exponent i.e exponential type of order two) envelope which can inflate arbitrarily fast.

Condition D, constant frequency, Gaussian amplitude

Consider

$$\beta = 0,\ \alpha \to -\infty \Rightarrow f(t) = \exp\left(\rho_0 + \rho_1 t - |\alpha| \rho_2 t^2\right) \cos\left(\varphi_0 + \varphi_1 t\right) \tag{18}$$

which is a fixed frequency signal with an arbitrarily fast Gaussian envelope drop-off.

Overall Conditions

Overall, we can say (unsurprisingly) that positivity can be maintained in extreme chirp situations if the amplitude is a Gaussian. When the frequency is constant, the exponential envelope has a wide range of possible second order forms.

Extension to 2-D Energy Tensor

What is analogous to positivity for the 2-D and higher dimensional [6]–[8] energy operators? Felsberg [9] suggests that the eigenvalues of the energy tensor must be positive. In other words the 2-D energy operator matrix must be positive semi-definite. Complete analysis must consider the interplay of five attenuation partial derivatives (up to second order) and five more partial derivatives of the phase. The positivity solution may be bounded by a manifold in a ten dimensional space. In a similar manner the positivity of the 1-D energy operator is bounded by a manifold in a four dimensional space, except that one dimension cancels out and another is folded into a normalized space, leaving the simple parabolic boundary in a 2-D space presented above. The proliferation of dimensions for 2-D energy operator positivity places it out of this paper's scope.

Discussion

Analysis of the Teager-Kaiser energy operator's properties is simplified by the logarithmic formulation; essentially the limits on phase modulation and amplitude modulation are intertwined by a parabolic relation of the normalized chirp-rate and the normalized attenuation curvature. The TKEO can remain positive for surprisingly fast attenuation rates, but in real band-limited signals the amplitude in such regions may be negligible.

Notes

Why is Teager-Kaiser Special?

One way to way to explain the special property of the Teager-Kaiser energy operator is encapsulated by Eq(6). It is that all first order errors in the estimate exactly cancel out leaving only (generally smaller) second order errors.

Quadrature functions

If the exact quadrature function is known, then demodulation/energy estimation is trivial using the amplitude and phase (and hence local frequency or phase derivative) directly from the argument of the analytic signal:

$$f(t) + i\hat{f}(t) = b(t).\cos\varphi(t) + ib(t).\sin\varphi(t) \Rightarrow \arg\left[f(t) + i\hat{f}(t)\right] = b(t) + i\varphi(t). \tag{1a}$$

Function space

Twice differentiable C^2 real functions.

Discrete formulation

Not considered here.

Harmonic Analysis

In an earlier publication [6] I claimed that positivity of the energy operator is equivalent to Hormander's [10] (p16-21) definition of subharmonicity, based on the Laplacian of the logarithm. Now it seems that I omitted a negative sign, which means positivity equates to superharmonicity instead – a very different implication.

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