# UNSW Research Workshop

*Function Spaces and Applications,*
*December 2-6, 2008.*

## Venue

Room 3085,
Red Centre Building,
School of Mathematics and Statistics,
University of New South Wales,
Kensington, 2052,
Sydney, NSW,
Australia.

## Schedule

| **Date**<br>**Time** | **Tuesday**<br>**02/12/08** | **Wednesday**<br>**03/12/08** | **Thursday**<br>**04/12/08** | **Friday**<br>**05/12/08** | **Saturday**<br>**06/12/08** |
|---|---|---|---|---|---|
| **9:00-10:00** | Dooley | Walter | Besov | Kashin | Leinert |
| **10:10-11:10** | Larkin | Taubman | Burenkov | Faierman | Meaney |
| **11:15-12:15** | Doust | Duong | Stepanov | Fletcher | lunch |
| **12:20-13:20** | Ignjatovic | lunch | lunch | lunch | excursion |
| **13:20-14:20** | lunch | McCoy | Wahlberg | Taggart | excursion |
| **14:25-15:25** | | Le Gia | Ajiev | Ji Li | excursion |

**Lie Symmetry Groups and Harmonic Analysis**

*Anthony H. Dooley*
School of Mathematics and Statistics, University of New South Wales

**Abstract** Sophus Lie showed how to calculate the Lie algebra of infinitesimal symmetries of a partial differential equation. These lead to local transformations of the $X \times U$-space which map solutions $y(x) = u$ to solutions. In work with Mark Craddock, I have studied these symmetries for one-dimensional heat kernels with drift. In many cases, the local symmetry groups may be extended to global actions, and act as global projective representations on the solution spaces.

I shall discuss some recent work where the group of symmetries is the semi-direct product of $SL(2, \mathbb{R})$ and the three dimensional Heisenberg group (the oscillator group), where explicit intertwining operators can be written down for its classical projective representations.

These ideas have been applied in option pricing.

**Which function spaces do fingerprints inhabit?**
**And why should we care?**

*Kieran G. Larkin*
Canon Information Systems Research Australia

**Abstract** Over the last 40 years many attempts have been made to find an effective mathematical representation for human fingerprint images. Most attempts have failed. The main reason for failure is clear in retrospect.

The most promising representations have been based on AM-FM models (Amplitude Modulation-Frequency Modulation). The concept of instantaneous frequency in two-dimensions is central to such models. Unfortunately the instantaneous frequency contains numerous singularities which cannot be easily contained within such a modulation model.

As an alternative we propose a phase-modulation model to tame the singularities. Once phase is chosen as the basis a very simple functional form can be defined for a fingerprint image:

$$f(\mathbf{r}) = a(\mathbf{r}) + b(\mathbf{r}) \cos[\Psi(\mathbf{r})0$$

It turns out that most of the information in a fingerprint image is contained in the phase term. An essential property of the phase is that it is not differentiable at all the important fingerprint matching points, also known as minutiae (or ridge endings and bifurcations). Although some researchers have recently recognised that fingerprint synthesis was possible using the above phase modulation model, it was not until 2006 that a corresponding analysis technique was announced to complete the process.

The heart of the analysis process consists of three main operations:

- isotropic demodulation using the Riesz transform

- isotropic direction estimation using a nonlocal energy operator combined with phase unwrapping
- unique phase separation using the Helmholtz-Hodge decomposition theorem:

$$\psi(\mathbf{r}) \equiv \psi_1(\mathbf{r}) + \psi_2(\mathbf{r})$$

The last step can be considered as a potential function representation of the underlying vector fields. It turns out that one phase component represents the curl-free field, whilst the other represents the divergence-free field. The former encodes the large scale structure of a fingerprint (whorls, loops, arches etc), the latter encodes minutiae - the quintessential matching features.

The phase modulation model splits an image into four elemental sub-images $a(\mathbf{r}), b(\mathbf{r}), \psi_1(\mathbf{r}), \psi_2(\mathbf{r})$. Each of the sub-images is sparse and can be represented by a small amount of (digital) information. Conventional (wavelet) methods of fingerprint image compression attain compression factors of about 15. The new method has been shown to provide at least an order of magnitude improvement (a compression factor of 200 or greater). An additional benefit is that fingerprint matching can occur in the compressed domain: the sparse data directly represents the significant visual features!

In this talk we will present the techniques required to make the fingerprint representation work, as well as showing some examples.

## Some problems concerning $AC(\sigma)$ operators

*Ian Doust*
University of New South Wales

**Abstract** There is a now classical area of spectral theory centred around Banach space operators which possess a functional calculus for the absolutely continuous functions on an interval of the real line $AC[a, b]$. Such operators necessarily have real spectrum.

In attempting to extend this theory to operators with complex spectrum, Brenden Ashton and I introduced two new Banach algebras $AC(\sigma)$ and $BV(\sigma)$ consisting of complex-valued absolutely continuous and bounded variation functions defined on a nonempty compact set $\sigma$ in the plane.

While we have been able to prove many of the types of spectral theoretic results that one might hope for in this setting, some quite simple questions still remain, and it is these open problems that I will discuss in this talk.

## Chromatic derivatives and associated expansions

*Aleksandar Ignjatovic*
University of New South Wales

**Abstract** We present a motivation for introducing the chromatic derivatives and the chromatic expansions, as well as their fundamental properties.

The chromatic derivatives are special, numerically robust linear differential operators. The chromatic expansions are the associated local expansions, which possess the best features of both the Taylor and the Nyquist expansions. These features make them potentially useful in a variety of fields which involve sampled data, such as digital signal and image processing. We also present some function spaces associated with the chromatic derivatives, including a non separable inner product space in which any two pure harmonic oscillations of distinct positive frequencies are mutually orthogonal.

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## Chromatic Series and Prolate Spheroidal Wave Functions

*Gilbert G. Walter*
University of Wisconsin-Milwaukee, USA

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**Abstract** Prolate spheroidal wave functions (PSWFs) arise as solutions of an integral equation. This makes them bandlimited functions in a Paley-Wiener space, but because they are also solutions to a Sturm-Liouville problem, they behave very much like polynomials locally.

Chromatic series are series in which the coefficients are linear combinations of derivatives of a function at a point. They were introduced by Ignjatovic as a replacement for Taylor's series and are based on orthogonal polynomials. Since the PSWFs are close to orthogonal polynomials they can be used to replace them in the Ignjatovic theory. The theory can be extended further to

prolate spheroidal wavelet series that then combine chromatic series with sampling series.

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## Motion and Geometry Adaptive Wavelet-Like Bases

*David S. Taubman*
University of New South Wales

**Abstract**

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## Function spaces associated with operators

*Xuan Duong*
Macquarie University

**Abstract** Let $L$ be a generator of an analytic semigroup with appropriate kernel bounds. We will define function spaces such as Hardy spaces, BMO spaces and Besov spaces associated with $L$ and compare them with the classical spaces.

We will also discuss boundedness of singular integrals on these spaces.

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## A new class of fully nonlinear curvature flows

*James McCoy*
University of Wollongong

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**Abstract** This talk is about contraction by fully nonlinear curvature flows of convex hypersurfaces. As with previously considered flows, including the quasilinear mean curvature flow and fully nonlinear Gauss curvature flow, solutions exist for a finite time and contract to a point. As with some other flows, including the mean curvature flow, under a suitable rescaling the solutions converge exponentially to spheres.

The main points of interest in this work are the allowance of nonsmooth initial data and that the only second derivative requirement on the speed is weaker than a requirement of convexity. We obtain new results in both cases of smooth and nonsmooth initial data.

This is joint work with Ben Andrews and Zheng Yu.

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## A spectral method with quadrature for Navier-Stokes equations on the rotating sphere

*Quôc Thông Lê Gia*
University of New South Wales

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**Abstract** Numerical algorithms for partial differential equations (PDEs) posed on the surface of spherical geometries play

an important role in addressing fundamental issues in various circulation and gravitational potential research models on the rotating Earth. In particular, the flows in the atmosphere are considered to be accurately modeled by the Navier-Stokes equations on the rotating sphere.

In this work, we describe, analyze, and implement a quadrature spectral method for a global computer modeling of the incompressible Navier-Stokes equations on the rotating unit sphere.

This is joint work with M.Ganesh (Colorado) and I.Sloan (UNSW).

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## Spaces of Functions of Fractional Smoothness on an Irregular Domain

*Oleg V. Besov*
Steklov Mathematical Institute, Moscow

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**Abstract** Function spaces on irregular domains of certain type of multidimensional Euclidean spaces are studied. Such domains may have, for instance, the shape of the external peak with a power-like degeneracy in the neighborhood of some boundary point. The embedding theorems for these spaces are established.

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**Sharp spectral stability estimates**
**for uniformly elliptic differential operators**

*Viktor Burenkov*
Cardiff University/University di Padova

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**Abstract** We consider the eigenvalue problem for the operator

$$Hu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left( A_{\alpha\beta}(x) D^\beta u \right), \quad x \in \Omega,$$

subject to homogeneous Dirichlet or Neumann boundary conditions, where $m \in \mathbb{N}$, $\Omega$ is a bounded open set in $\mathbb{R}^N$ and the coefficients $A_{\alpha\beta}$ are real-valued Lipschitz continuous functions satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \theta|\xi|^2$$

for all $x \in \Omega$ and for all $\xi_\alpha \in \mathbb{R}, |\alpha| = m$, where $\theta > 0$ is the ellipticity constant.

We consider open sets $\Omega$ for which the spectrum is discrete and can be represented by means of a non-decreasing sequence of non-negative eigenvalues of finite multiplicity $\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \ldots$ Here each eigenvalue is repeated as many times as its multiplicity and $\lim_{n\to\infty} \lambda_n[\Omega] = \infty$.

The aim is sharp estimates for the variation $|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$ of the eigenvalues corresponding to two open sets $\Omega_1$, $\Omega_2$ with continuous boundaries, described by means of the same *fixed* atlas $\mathcal{A}$.

There is vast literature on spectral stability problems for elliptic operators. However, very little attention has been devoted

to the problem of spectral stability for higher order operators and in particular to the problem of finding explicit qualified estimates for the variation of the eigenvalues. Moreover, most of the existing qualified estimates for second order operators were obtained under certain regularity assumptions on the boundaries.

Our analysis comprehends *operators of arbitrary even order*, *with homogeneous Dirichlet or Neumann boundary conditions*, and *open sets admitting arbitrarily strong degeneration*.

Three types of estimates will be under discussion: for each $n \in \mathbb{N}$ for some $c_n > 0$ depending only on $n$, $\mathcal{A}, m, \theta$ and the Lipschitz constant $L$ of the coefficients $A_{\alpha\beta}$

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n d_{\mathcal{A}}(\Omega_1, \Omega_2),$$

where $d_{\mathcal{A}}(\Omega_1, \Omega_2)$ is the so-called *atlas distance* of $\Omega_1$ to $\Omega_2$,

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)),$$

where $d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)$ is the so-called *lower Hausdoff-Pompeiu deviation* of the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ and $\omega$ is the common modulus of continuity of $\partial\Omega_1$ and $\partial\Omega_2$, and, under certain regularity assumptions on $\partial\Omega_1$ and $\partial\Omega_2$,

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \mathrm{meas}\left(\Omega_1 \Delta \Omega_2\right),$$

where $\Omega_1 \Delta \Omega_2$ is the symmetric difference of $\Omega_1$ and $\Omega_2$.

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**Two-sided estimates of the approximation and entropy numbers**

**of the integral operators**

*Vladimir D. Stepanov*
Peoples' Friendship University, Moscow

**Abstract** Important characteristics of the compactness (or measure of non-compactness) of a linear operator $T : E \to F$, acting from the Banach space $E$ to the Banach space $F$, are the approximation and entropy numbers, $a_n(T)$ and $e_n(T)$, defined as follows:

$$a_n(T) := \inf\{\|T-S\|_{E\to F} : \ S : E \to F, \ \operatorname{rank}(S) \le n-1\}, \quad n \in \mathbb{N},$$

$$e_n(T) := \inf\Big\{\varepsilon > 0 \ : \ T(B_E) \subseteq \bigcup_{j=1}^{2^{n-1}} B_F(Tx_j;\varepsilon), x_1, \ldots x_{2^{n-1}} \in B_E\Big\},$$

$n \in \mathbb{N}$, where

$$B_E := \{x \in E : \|x\|_E \le 1\}, \quad B_F(y,\varepsilon) := \{x \in F : \|x-y\|_F < \varepsilon\}$$

and $\mathbb{N}$ denotes the set of all natural integers.

Wide bibliography is devoted to investigation of the above and some other characteristic numbers of linear operators. The cases of a particular interest are those, when two-sided sharp estimates exist for the sequence $\{a_n(T)\}$ and $\{e_n(T)\}$.

During the last decade essential attention was paid to the study of the $a$-numbers and $e$-numbers of the Hardy and Volterra integral operators. The communication gives a short survey of new results in this area.

**Gabor discretization of the Weyl product acting on modulation spaces**

*Patrik Wahlberg*
University of Newcastle

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**Abstract** We discuss the Weyl quantization of pseudodifferential operators and the Weyl product $(a\#b)^w = a^w \circ b^w$. There is a continuity result for the Weyl product acting on weighted modulation spaces $M_m^{p,q}(\mathbb{R}^d)$ of the form

$$\|a_1 \# a_2\|_{M_{m_0}^{p_0,q_0}} \leq C \|a_1\|_{M_{m_1}^{p_1,q_1}} \|a_2\|_{M_{m_2}^{p_2,q_2}}$$

for certain weights $m_j$ and exponents $p_j, q_j \in [1, \infty]$.

We discretize $a_0 = a_1 \# a_2$ using a Gabor frame defined by a Gaussian function. The main result is: If $a_2 \in M_\omega^{\infty,1}$ then the matrix of $a_1 \mapsto a_1 \# a_2$ decays off the diagonal. If $\omega(X,Y) = \exp(s|Y|)$, where $s \geq s_0$ and $s_0$ is sufficiently small, then the decay is exponential. If $\omega(X,Y) = (1+|Y|^2)^{s/2}$, $s > 2d$, then the decay is polynomial. Finally we apply the results to the problem of filtering (denoising) second-order nonstationary stochastic processes.

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*Sergey Ajiev*
University of New South Wales

**Abstract**

## Compressed Sensing and Error-correcting Codes

*Boris S. Kashin*
Steklov Mathematical Institute, Moscow

**Abstract**

## On the Essential Spectrum of an Operator Arising in Magnetohydrodynamics

*Melvin Faierman*
University of New South Wales

**Abstract** We consider a problem introduced by Descloux and Geymonat in 2-dimensional magnetohydrodynamics wherein all coefficients involved depend only upon one of the space variables. Because of this, we show how it is possible to completely characterize the essential spectrum of the induced Hilbert space operator by reducing the problem to one studied by Gohberg and Krein concerning systems of integral equations.

## Hiding Hyperbolic Harmonics

*Peter Fletcher*
Canon Information Systems Research Australia

**Abstract** A problem that has received a lot of attention in the last decade is digital image watermarking, which is the process of embedding and detecting secret, robust marks in a carrier image. A good watermark is imperceptible to viewers, easy and reliable to detect using a computer, and robust to image distortions, both malicious and accidental. This is similar to steganography; a more general application where secret marks are embedded in a carrier image, but are not required to be robust.

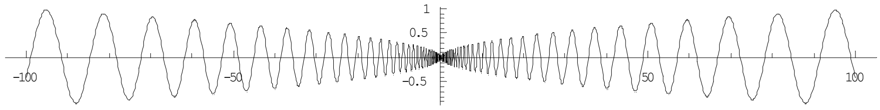
The first suggested marks for watermarking were pseudo-random noise, which are imperceptible if embedded at low levels in an image; easy to detect using correlation; robust to image cropping; and very difficult for an attacker to remove without knowledge of the mark. Unfortunately, the detection of noise patterns is not robust to common image processing techniques such as scaling and rotation.

Other marks which have been suggested are repetitive tilings of noise patterns, which are detectable through auto-correlation, and groups of sinusoids, which are detectable as distinct peaks in the Fourier transform. These marks can be detected even after affine distortion of the carrier. Unfortunately, the ease of detection of these methods makes them an easy target for malicious attack, as with knowledge of the watermarking method

it is possible for an attacker to identify and remove the marks with little difficulty.

We have developed watermarks which can be embedded imperceptibly into an image, are robust to affine distortion and cropping, are easy to detect with knowledge of the mark, and moderately difficult to detect without knowledge of the mark. These marks are based on the space of logarithmic harmonic functions (LHFs), also known as hyperbolic chirps, which in one dimension have the form

$$f(x) = \exp[(a + ib)\ln|x|] \equiv |x|^{a+ib}$$




These functions are complex and have the convenient property that their frequency is inversely proportional to the distance from the origin. This leads to the remarkable property that

$$f(\sigma x) \equiv |\sigma|^{a+ib} f(x)$$

i.e. the function when scaled is invariant up to a complex multiplier. The function is in fact homogeneous of complex degree $a + ib$. These functions have the remarkable, but little known spread-spectrum property (indeed, the Fourier transform of an LHF is itself an LHF) they are easily detectable by correlation, and thus make good candidates for watermarks.

It is generally difficult to embed a complex function in an image, but fortunately the real part of an LHF shares many of its properties. To use LHFs as watermarks in a two-dimensional image, they can be back-projected to form a two-dimensional image. The resulting function is quite limited spectrally, as it exists only in a single line of the Fourier transform. However, this does not pose much of a problem in terms of embedding and detection, because the function is still limited in perceptibility, and can be concentrated and detected by using a Radon transform followed by a one dimensional correlation.

The scaling property of the one-dimensional function is preserved in its projection, so any affine transform applied to an image containing an LHF will result in an image containing the same LHF with a possible rotation and phase shift.

This talk will present many interesting properties of LHFs, and show how they can be used when creating a practical image watermarking system.

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## Strichartz estimates for the wave equation and Schrodinger equation with potential

*Robert Taggart*
Australian National University

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**Abstract** Strichartz estimates represent one of the most useful tools for the analysis of dispersive linear and nonlinear PDEs. After a brief introduction to Strichartz estimates, this talk will focus primarily on new Strichartz estimates for the wave equation (where regularity in the spatial variable is measured using

homogeneous Besov norms) and for Schrodinger's equation with a certain class of potentials.

These results are derived from the abstract Strichartz estimates of Markus Keel and Terry Tao and from those of the speaker which extend their work.

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**Duality of Hardy spaces $H^p$ on product spaces of homogeneous type and applications.**

*Ji Li*
Macquarie University/ Sun Yat-sen University, China

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**Abstract** In this paper, we introduce the Carleson measure space $CMO^p$ on product spaces of homogeneous type in the sense of Coifman and Weiss , and prove that it is the dual space of the product Hardy space $H^p$ of two homogeneous spaces defined by Y.S. Han, G.Z. Lu and D.C. Yang. Our results thus extend the duality theory of Chang and R. Fefferman on $H1(R^2_+ \times R^2_+)$ with $BMO(R^2_+ \times R^2_+)$ which was established using bi-Hilbert transform. Our method is to use discrete Littlewood-Paley-Stein theory in product spaces. Since our method do not rely on Journe's covering lemma, we can apply it to other cases, such as product spaces of $n$-factors, manifolds, Lie-groups and so on.

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## Convolution dominated operators on groups

*Michael Leinert*
Ruprecht-Karls-Universität, Heidelberg

**Abstract** Let G be a discrete group and H the Hilbert space of square summable functions on $G$. An operator $T$ in the algebra $B(H)$ of all bounded operators on $H$ is called convolution dominated, if its matrix with respect to the canonical basis of $H$ is dominated by the matrix of the convolution with an absolutely summable function.

The algebra $CD(H)$ of all convolution dominated operators on $H$ is spectral in $B(H)$. For a commutative G this is due to Baskakov, Gohberg and al., and Kurbatov. The noncommutative case requires different methods, however. In the case of nondiscrete groups, the situation is more complicated, but still some results are possible.

This is joint work with G.Fendler and K.Groechenig.

## Almost everywhere convergence of spectral sums on Lie groups

*Christopher Meaney*
Macquarie University

**Abstract** This talk is based on joint work with Leonardo Colzani, Detlef Müller, and Elena Prestini.

Let Ł be a right-invariant sub-Laplacian on a connected Lie group $G$, and let $S_R f := \int_0^R dE_\lambda f,\ R \geq 0$, denote the associated "spherical partial sums," where Ł $= \int_0^\infty \lambda \, dE_\lambda$ is the spectral resolution of Ł. We prove that $S_R f(x)$ converges a.e. to $f(x)$ as $R \to \infty$ under the assumption $\log(2 + \text{Ł}) f \in L_2(G)$.