Warping of Bandlimited Images:

The Search for Perfection

Reconstructed presentation sequence with additional comments 8 April 2026

Warping of Bandlimited Images:

The Search for Perfection

Kieran G. Larkin
Nontrivialzeros Research
Bracewell @70 Meeting,
8-9 April 2026
Macquarie University,
Sydney, Australia

The Seven Deadly Sincs...

A MARGINAL FOOTNOTE IN THE HISTORY OF SAMPLING THEORY...

Subjects to be covered

Perfect Fourier warping for affine geometric warps

Near-perfect Fourier warping for non-affine warps

Ron Bracewell's response...

Asymptotic Perfection...

Surely it's all been done before?

- Yes….

persistent misunderstandings 1995 – 2004 plague the literature...

- but….

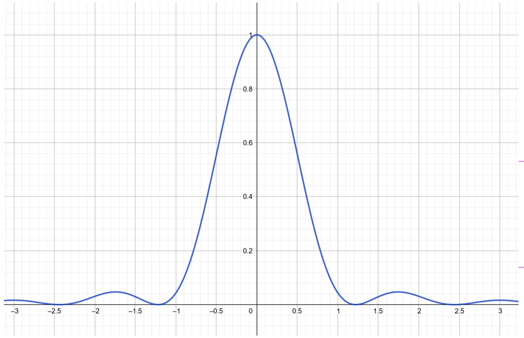
….this time correctly….

- Or is it too late

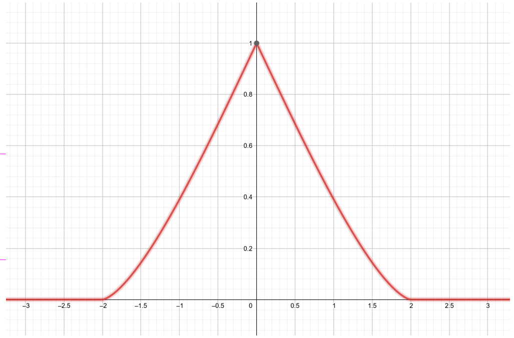
(e.g. AI Upscalers…)

Classic Band-Limited Imaging Systems

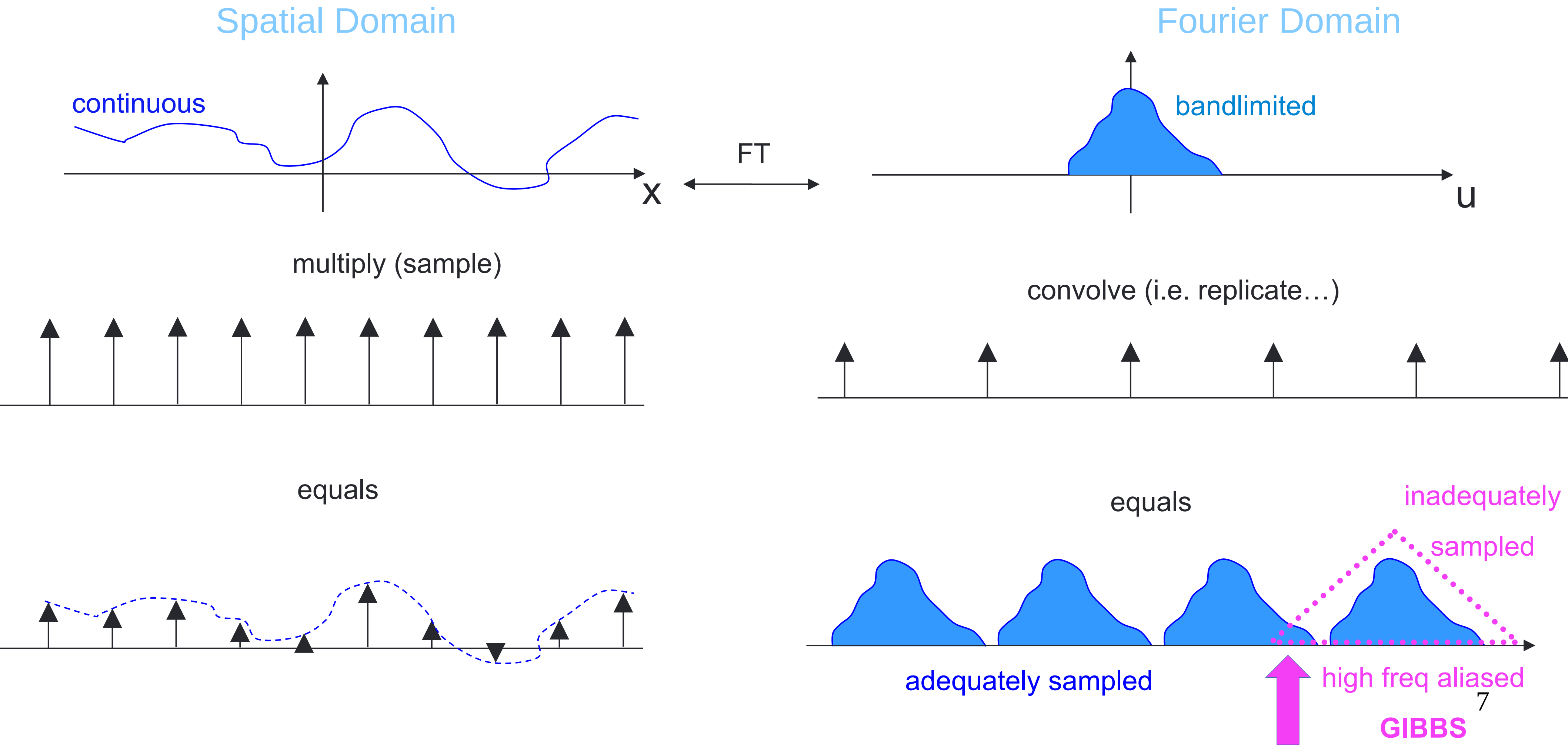
- Camera
- Human eye
- Telescope
- Microscope



PSF $p(x,y) \overset{FT}{\rightleftharpoons} P(u,v)$ OTF



Bandlimited Sampling 101 (reminder...)



Bandlimited Reconstruction 101

- Whittaker/Shannon/Kotelnikov/Ogura etc sampling-reconstruction
- Convolve with sinc (spatial domain)
- Or multipliply by rect (Fourier domain)

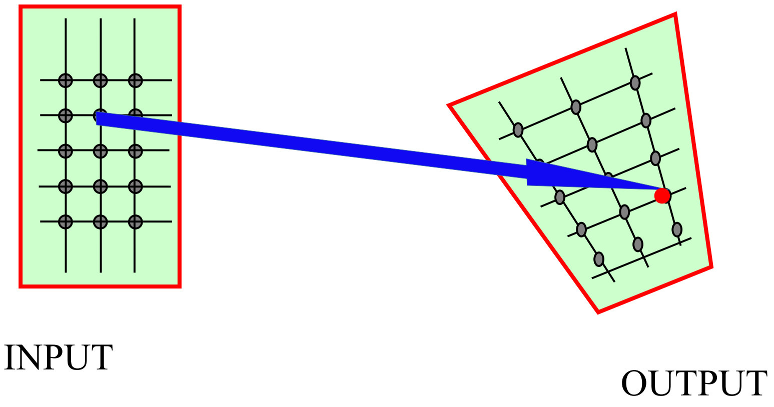
Central Premiss 1

Perfect Sinc Interpolation can be implemented
via
Fourier Shift and/or **Chirp Zoom**

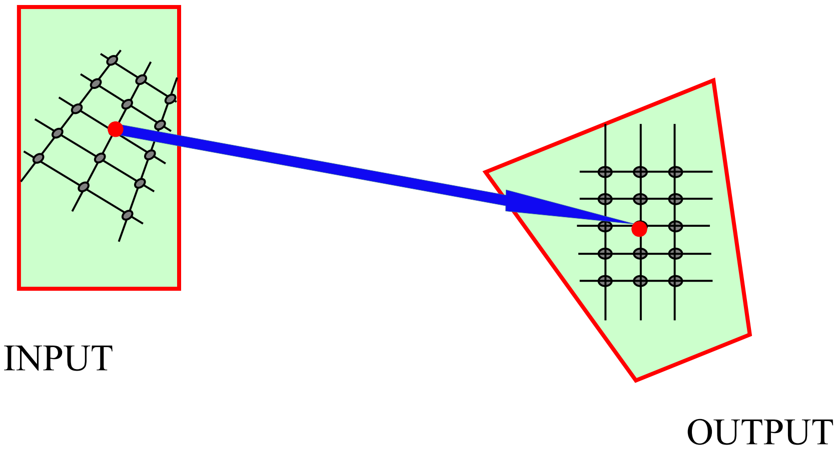
Hence

Hence FFT speed of order $N\log_2 N$ possible for affine geometric warps

Forward and Inverse Mapping

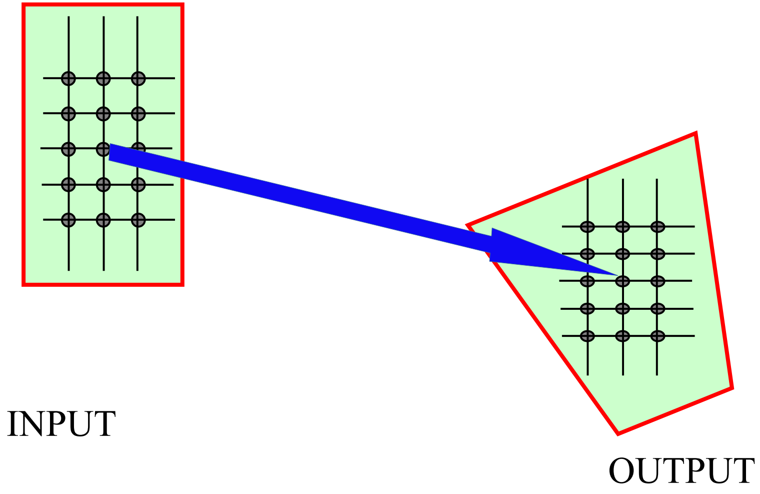


Forward mapping
From regular grid
To warped grid (interp kernel applied)



Inverse mapping
From inverse warped grid (interp kernel)
To regular grid

Proposed Mapping!



4D i.e. N^4 complexity

2D (N^2) array of
2D images (N^2 pixels)

No **explicit** interpolation kernel

Only complex multiplication on regular grids

Forward mapping
From regular grid
To regular grid

Diversion ahead

A general affine warping theorem for 2D FT

Bracewell
Electronics Letters 1993

AFFINE THEOREM FOR TWO-DIMENSIONAL FOURIER TRANSFORM

R. N. Bracewell, K.-Y. Chang, A. K. Jha and Y.-H. Wang

Indexing terms: Fourier transform, Transforms

The well known shift and similarity theorems for the Fourier transform generalise to two dimensions but new theorems come into existence in two dimensions. Simple theorems for rotation and shear distortion are examples. A theorem is presented which determines what the Fourier transform becomes when the function domain is subjected to an affine co-ordinate transformation. The full theorem contains a variety of simpler theorems as special cases. It may prove useful in its general form in image processing where sequences of affine transformations are applied.

The affine theorem which we shall derive is:

If $f(x, y)$ has 2-D FT $F(u, v)$, then $g(x, y) = f(ax + by + c, dx + ey + f)$ has 2-D FT

$$G(u, v) = \frac{1}{|\Delta|} \exp\left\{\frac{i2\pi}{\Delta}[(ec - bf)u + (af - cd)v]\right\} \times F\left(\frac{eu - dv}{\Delta}, \frac{-bu + av}{\Delta}\right)$$

where the determinant Δ is given by

$$\Delta = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

To derive this result, express the affine co-ordinate transformation

$$x' = ax + by + c \qquad y' = dx + ey + f$$

in matrix notation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

and note the Jacobian relation $dx'\,dy' = |\Delta|\,dx\,dy$. If $\Delta \neq 0$, invert the transformation to obtain

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} x' - c \\ y' - f \end{bmatrix}$$

In the phase exponent $-i2\pi(ux + vy)$ that occurs in the definition of the two-dimensional Fourier component, note that

$$\begin{aligned} ux + vy &= [u \quad v]\begin{bmatrix} x \\ y \end{bmatrix} \\ &= [u \quad v]\begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} x' - c \\ y' - f \end{bmatrix} \\ &= \frac{1}{\Delta}[u \quad v]\begin{bmatrix} e & -b \\ -d & a \end{bmatrix}\begin{bmatrix} x' - c \\ y' - f \end{bmatrix} \\ &= \frac{1}{\Delta}[eu - dv \quad -bu + av]\begin{bmatrix} x' \\ y' \end{bmatrix} \\ &\quad - \frac{1}{\Delta}[eu - dv \quad -bu + av]\begin{bmatrix} c \\ f \end{bmatrix} \end{aligned}$$

Hence

$$G(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax + by + c, dx + ey + f) \times e^{-2\pi(ux + vy)}\,dx\,dy$$

$$= e^{i(2\pi/\Delta)[(ec - bf)u + (af - cd)v]} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y') \times e^{-i(2\pi/\Delta)[(eu - dv)x' + (-bu + av)y']}\,dx'\,dy'/|\Delta|$$

This completes the derivation. The expression can be condensed by referring to affine transform plane co-ordinates defined by $u' = (eu - dv)/\Delta$ and $v' = (-bu + av)/\Delta$; then

$$G(u, v) = \frac{1}{|\Delta|} e^{i(2\pi/\Delta)(cu' + fv')} F(u', v')$$

The inverse transformation to (u, v) co-ordinates is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & d \\ b & e \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

8th December 1992

R. N. Bracewell, K.-Y. Chang, A. K. Jha and Y.-H. Wang (*Electrical Engineering Department, Stanford University, Stanford, CA 94305-4055, USA*)

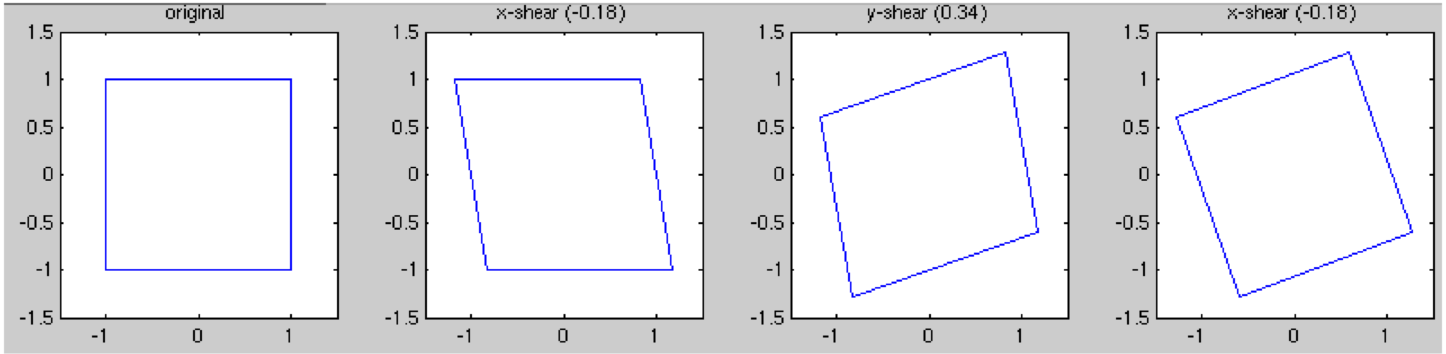
Affine Decompositions

General 2D affine:

- rotation, anamorphic scale, rotation

Rotation:

- **x-shear, y-shear, x-shear (Paeth 1986)**
- **y-shear, x-shear, y-shear**
- **reflect in axis1, reflection in axis 2 (Tong-Cox 1999)**



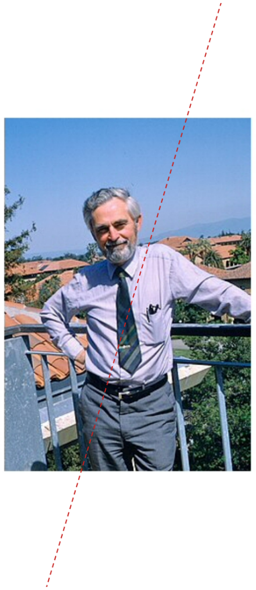
rotate=reflectx2

Some Uses of Cylindrical Lens-Systems, Including Rotation of Images

George J. Burch, Proc Roy Soc Lond, Vol. 73, (1904)

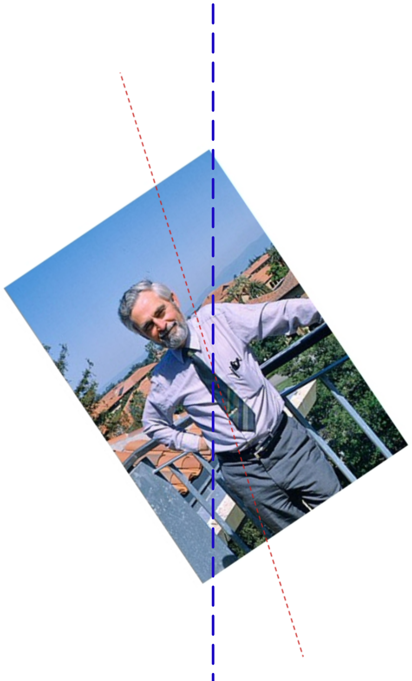
Two and three dimensional image rotation using the FFT

R. W. Cox and R. Tong, "IEEE Trans Im Proc 8, (1999)



Tilted axis -dotted line

Reflect in tilted axis



Reflect in vertical axis

overall rotation = twice angle between axes

Aliasing...3 shear...(45º worst case)

Spatial domain edge wraparound
Fourier domain---aliasing (Larkin,Opt.Comm. 1997)

Central Premiss 2

Edge and boundary truncation

Obliterates Bandlimitedness

and hence
must be properly
addressed…*before* resampling

source of endless misunderstandings….
~1992 – 2002
5000+ citations

Non Band-limited Image Structures

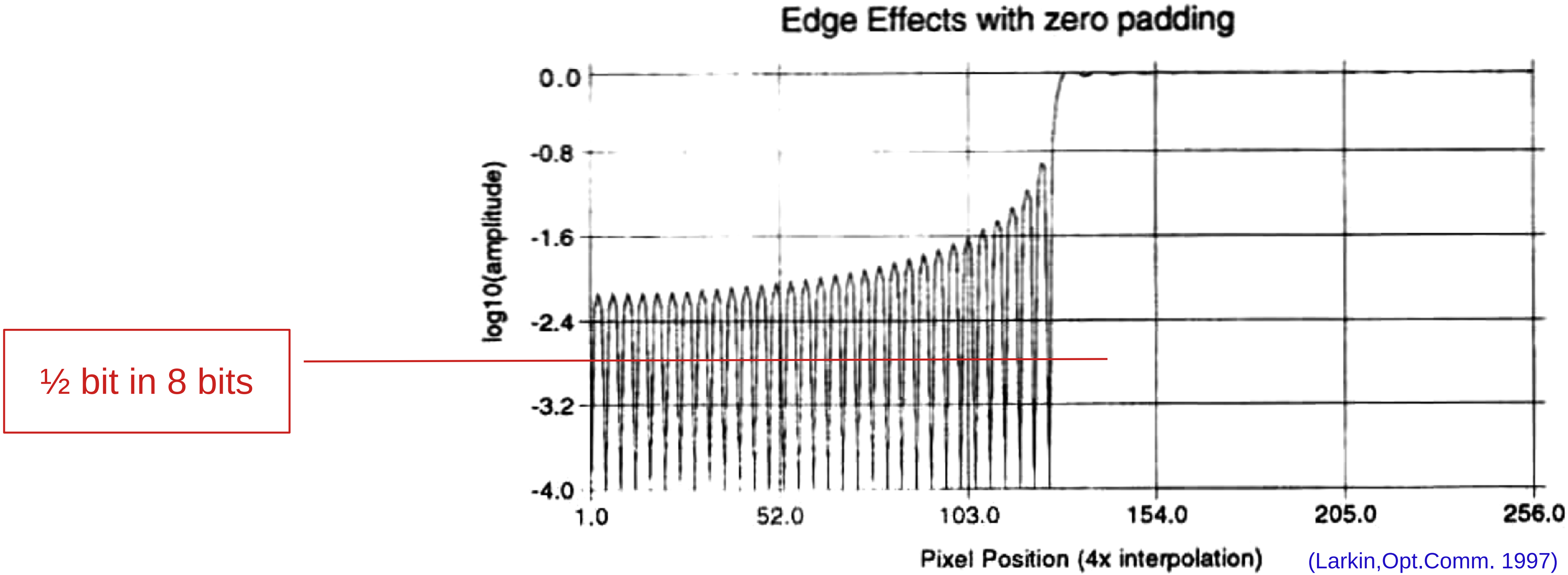
Image Sensor

Focal plane Aperture (photo frame)

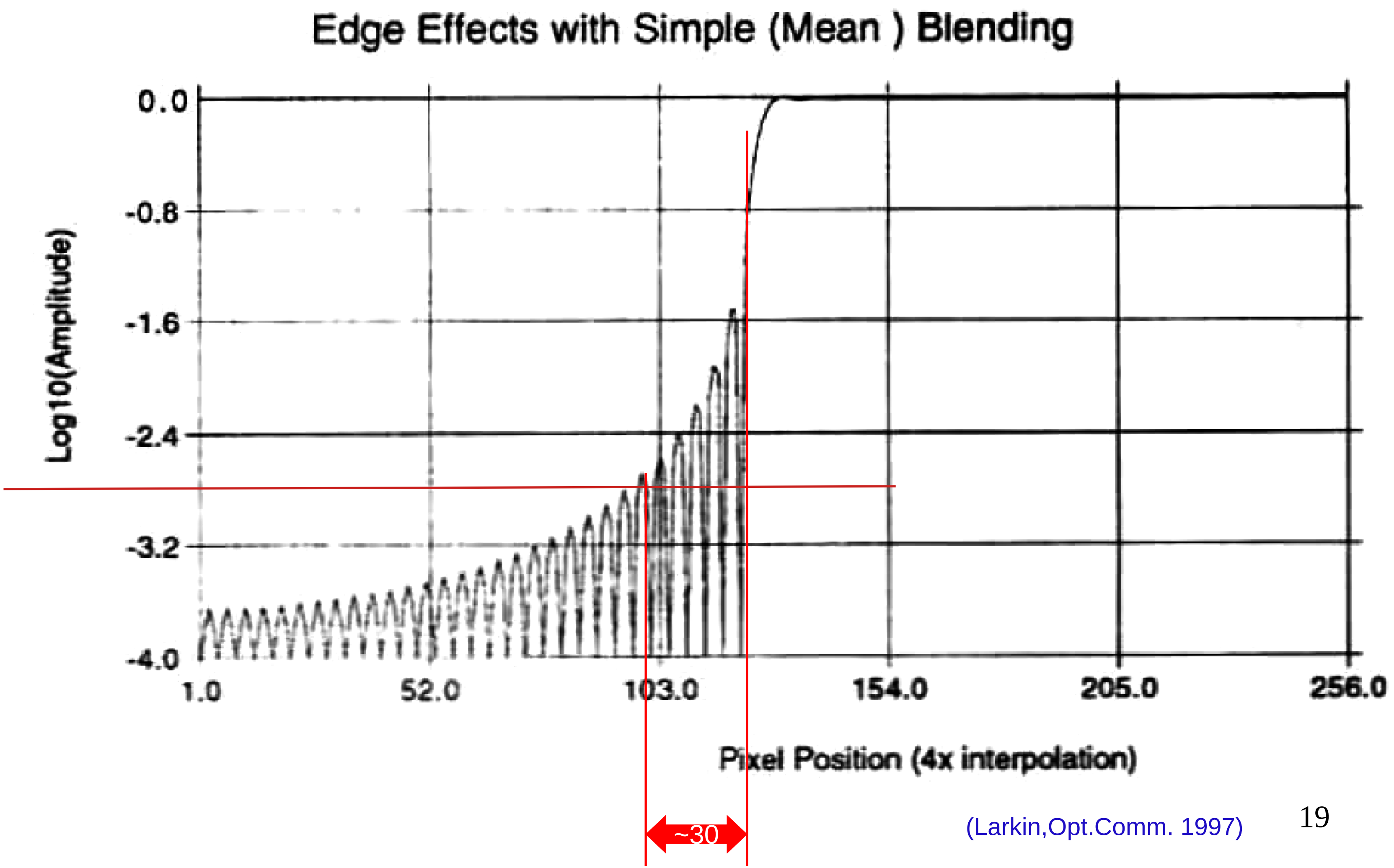
Telescope/microscope
tube/struts/stops/baffles

Edge Truncation Errors

- Truncation is the origin of **all** problems with image edge ringing (Gibbs phenomena)



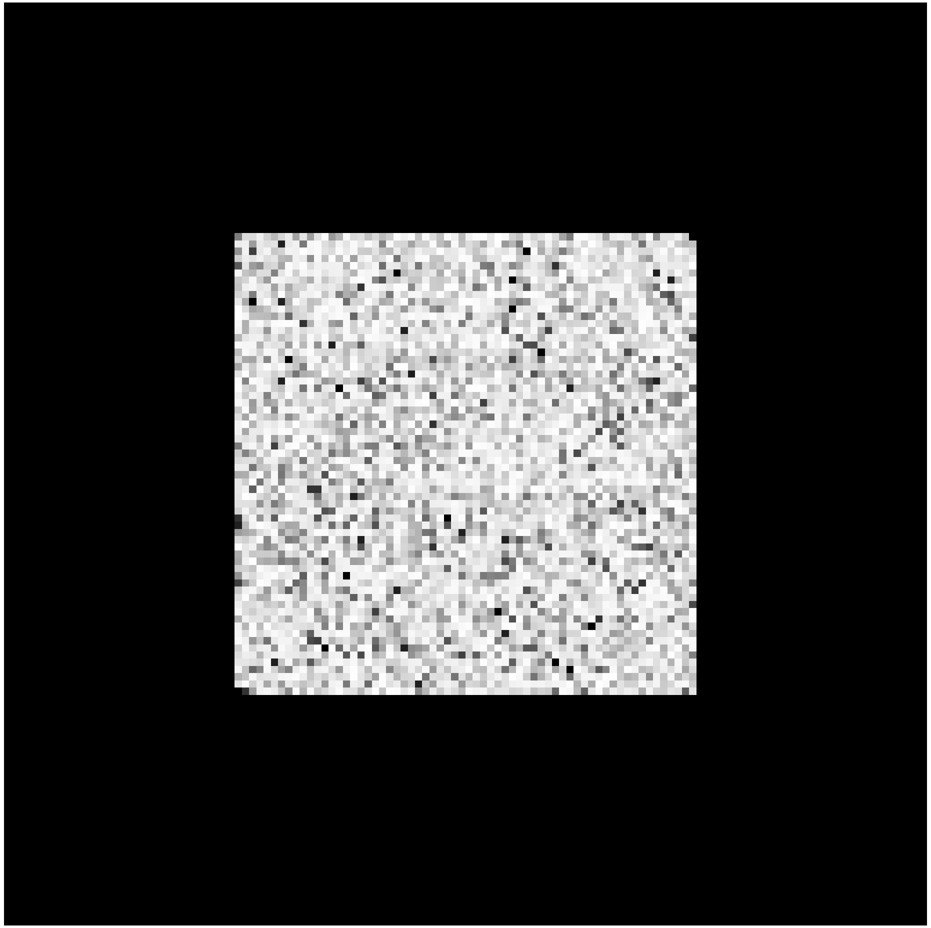
(Larkin,Opt.Comm. 1997)



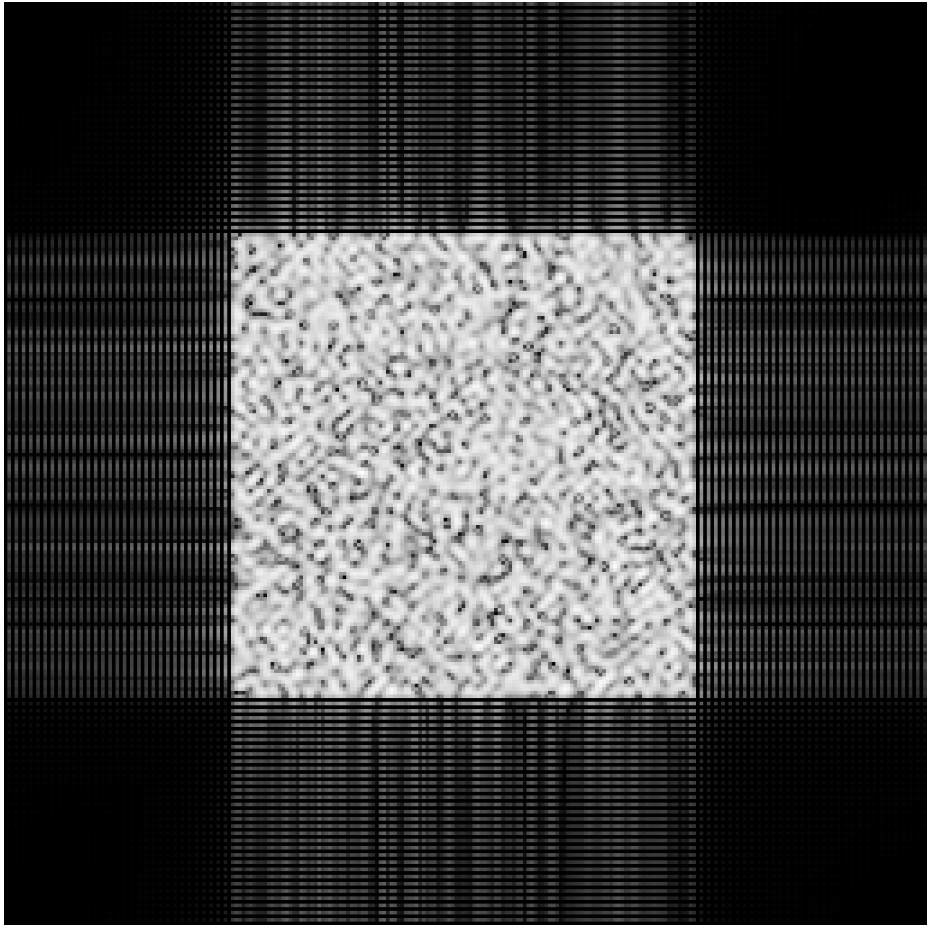
½ bit in 8 bits

(Larkin,Opt.Comm. 1997)

Bandlimited 64x64 noise image
Log greyscale display



Perfect upsample x2
Log greyscale display

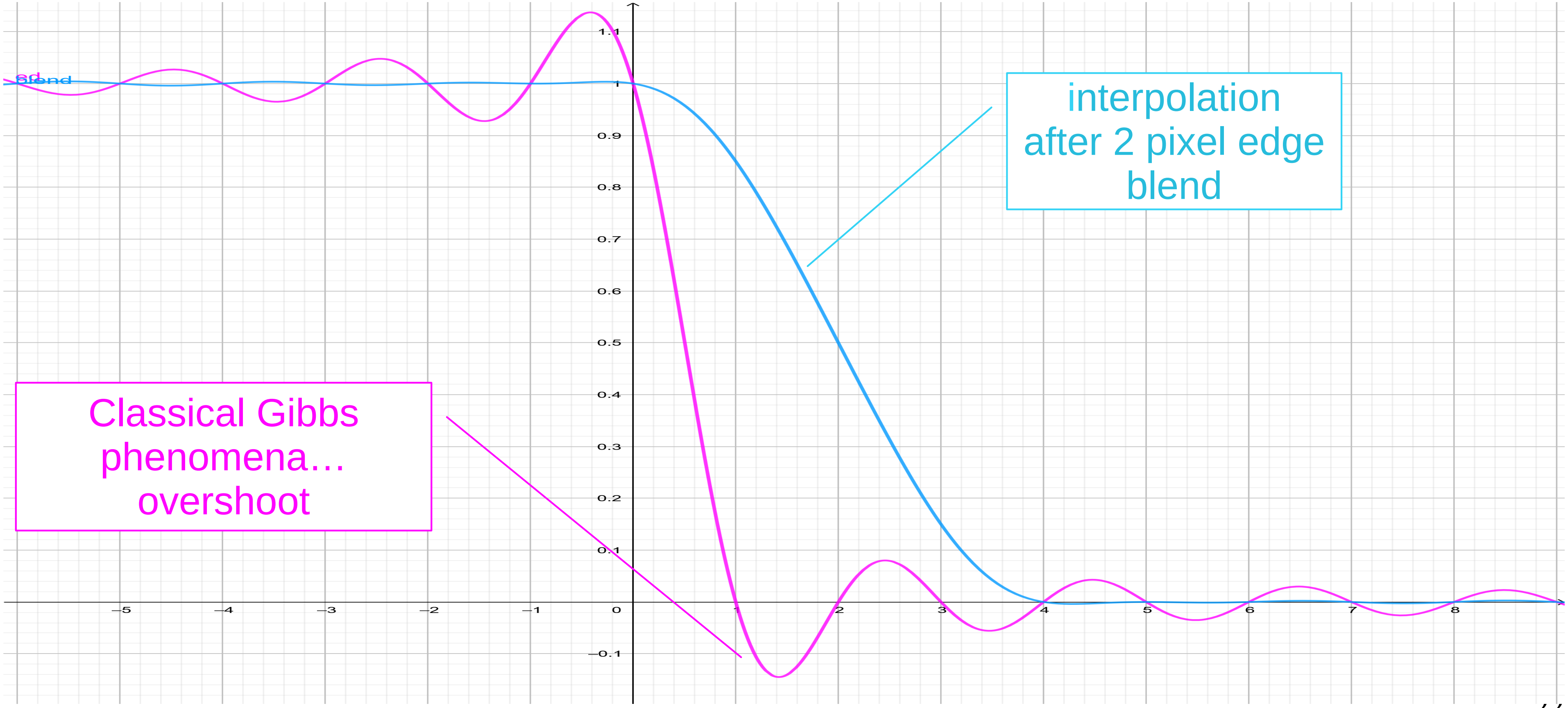


Solution: edge blend to zero

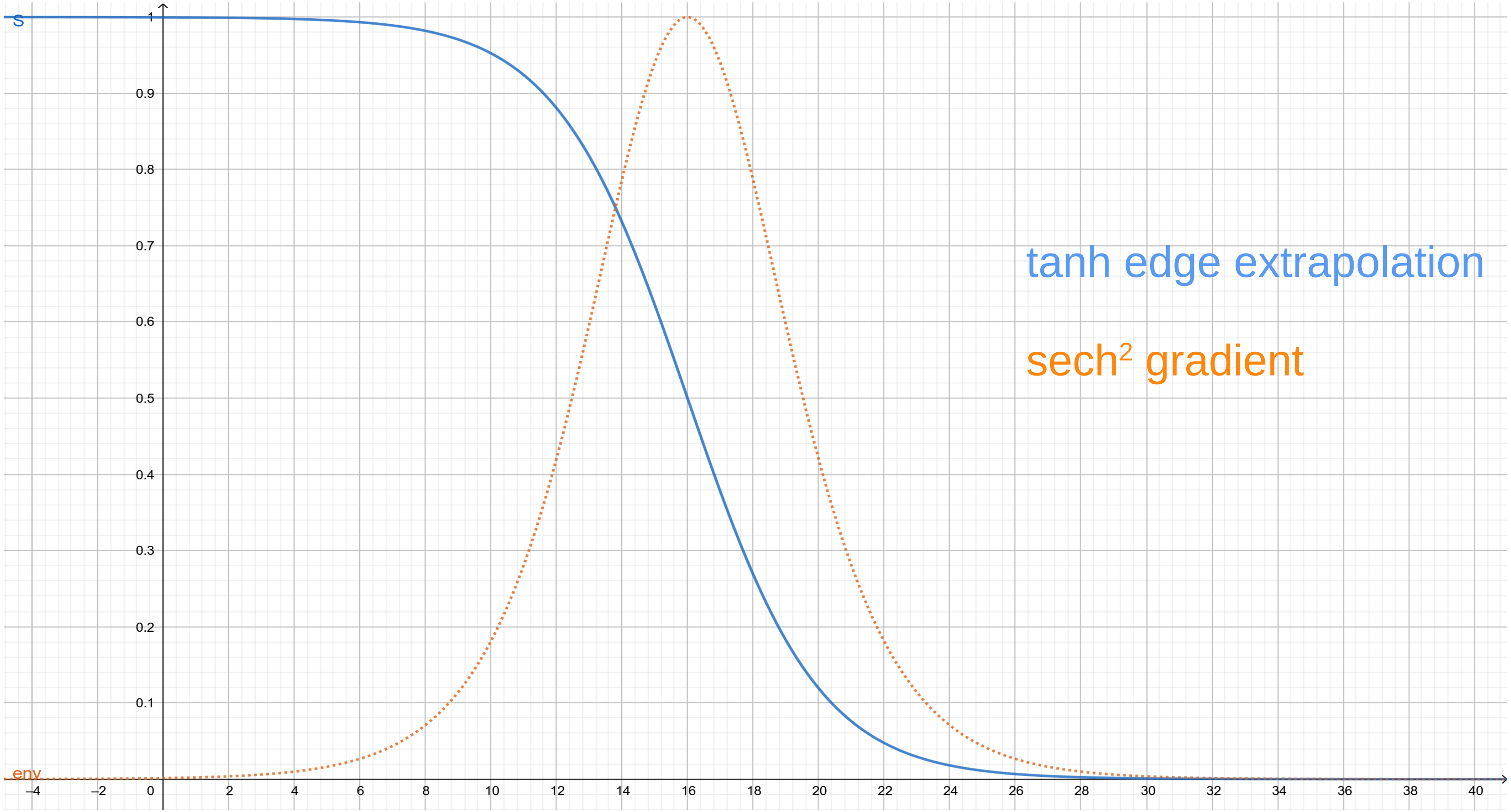
- **Ideally** PSWF (prolate spheroidal wave functions)
  - Only extends by a few pixels (Roy Frieden, Prog Opt 1971)
  - Implement via Gerchberg-Saxton iteration (keeps your GPU nice and warm)

- **Practically** direction blur/blend to zero
  - How far for 8 bit signals?
  - How far for 16 bit signals?

2 pixel edge blend



Proposed 32 pixel edge blend...



Central Premiss 3

Cardinal B-Splines (CBSs)

Which are **NOT** B-splines!

Very closely approximated by...

Hyperbolic "*sinch*" kernel
(KGL unpublished proof via 1992 Aldroubi-Unser-Eden Fourier CBS Formula)

Misleading...misinterpolation....ieee

Interpolation revisited [medical images application]	Philippe et al.	2000	IEEE Transactions on medical imaging
Linear interpolation revitalized	Blu et al.	2004	IEEE Transactions on Image Processing
Image Interpolation and Resampling	Blu et al.	2000	Handbook of medical imaging processing
Convolution-based interpolation for fast, high-quality rotation of images	Unser et al.	1995	IEEE Transactions on Image Processing

Allegedly finite CBSs replace “infinite length” sinc
But easily show CBSs, n>2 are also infinite
Actual...
Dirichlet kernel is periodic sinc--→sampled systems

Aldroubi Fourier Formulation

- Cardinal B-Spline h(x) spatial and H(u) Fourier domains

$$H_n(u) = \int_{-\infty}^{+\infty} h_n(x) \exp[-2\pi iux] dx$$

$$h_n(x) = \int_{-\infty}^{+\infty} H_n(u) \exp[+2\pi iux] du$$

- FIR/IIR Filter...direct FT

$$H_n(u) = \frac{\operatorname{sinc}^{n+1}(\pi u)}{\sum_{k=1}^{\infty} \operatorname{sinc}^{n+1}(\pi[u-k])}$$

an infinite number of infinite discontinuities...cancel out...only just...see Hurwitz zeta

Aldroubi Fourier Edge Sharpness

- Edge gradient controls cut-off

$$\frac{d}{du}H_n(u) \cong \operatorname{sech}^2\left(2[n+1][u+1/2]\right) - \operatorname{sech}^2\left(2[n+1][u-1/2]\right)$$

Outline Proof

Aldroubi Fourier form H(u)
Interpolation kernel h(x)

$$\mathrm{h}(x) \underset{}{\overset{FT}{\rightleftharpoons}} H(u)$$

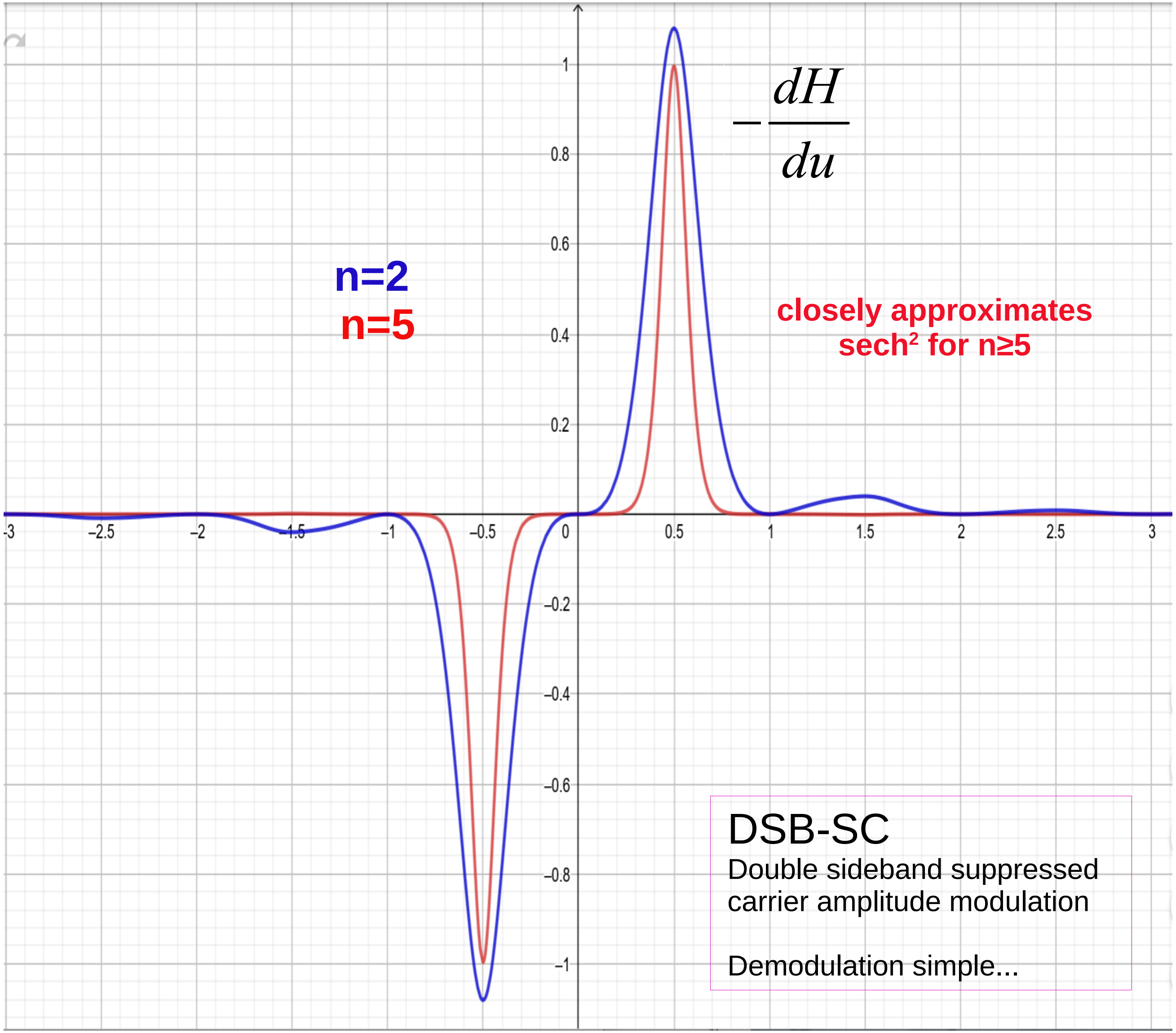
$$2\pi i x \mathrm{h}(x) \underset{}{\overset{FT}{\rightleftharpoons}} \frac{d}{du} H(u)$$

If h(x) =w(x).sinc(x)

Then

x.h(x) = w(x).sin(πx)

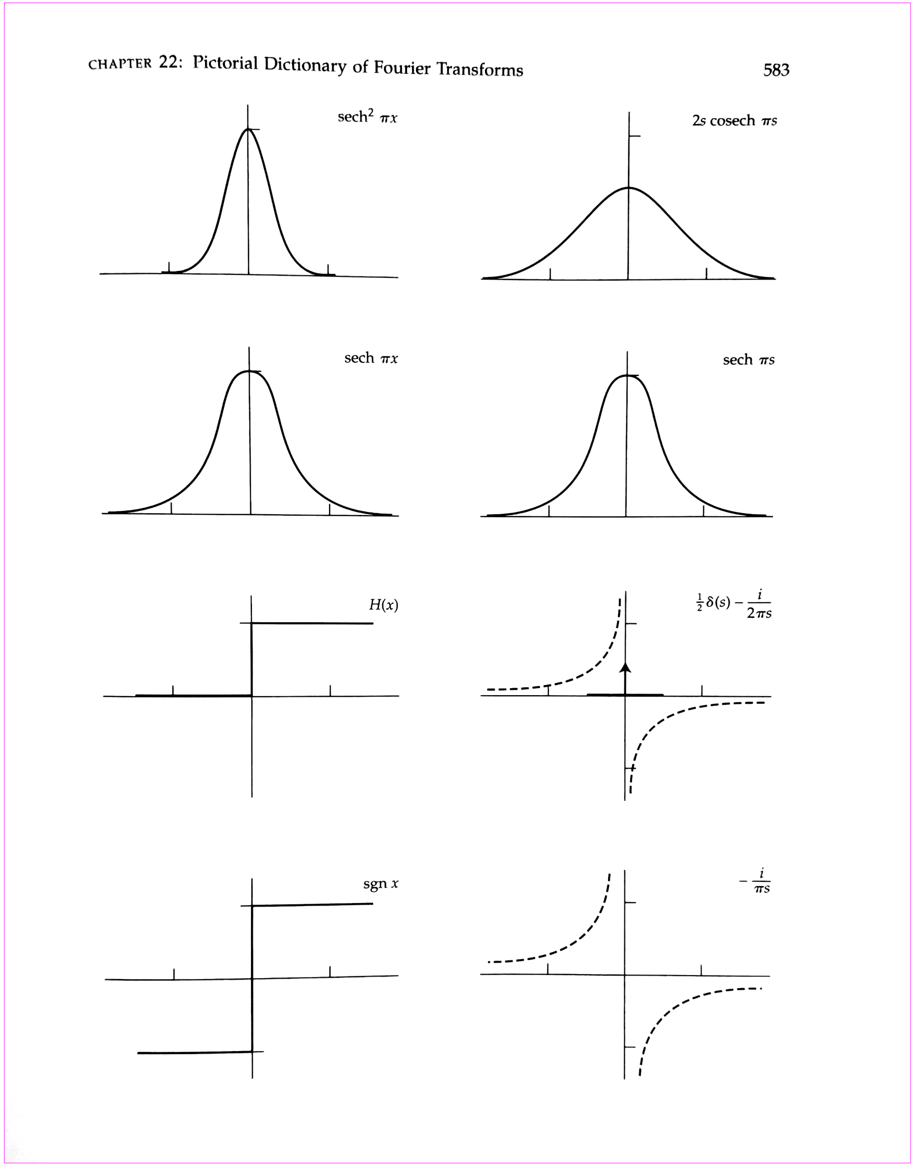
AM signal
=> modulation/demod. theory



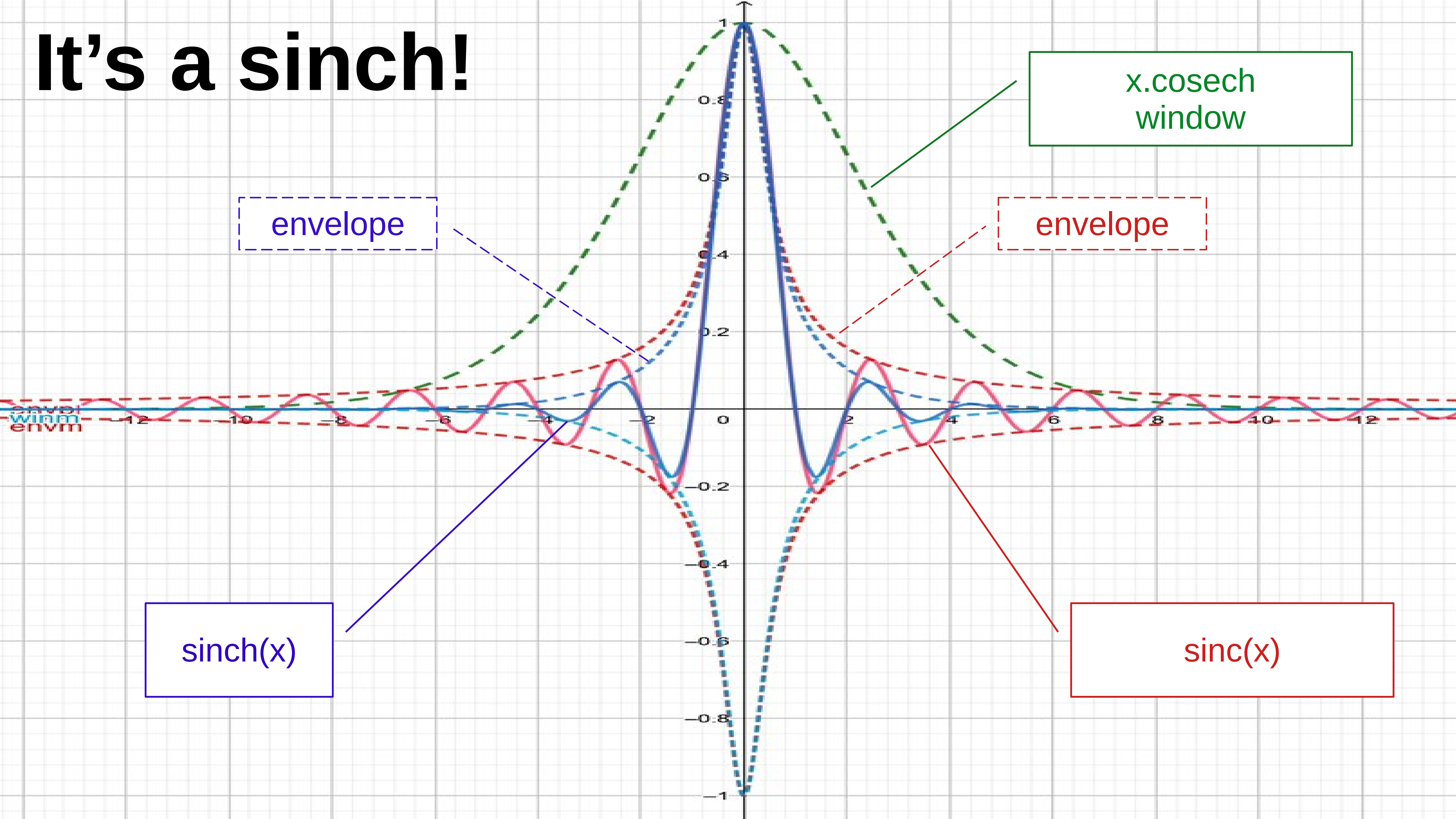
DSB-SC
Double sideband suppressed carrier amplitude modulation

Demodulation simple...

Bracewell FTA Pictorial



It’s a sinch!



Sinc versus Sinch

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)}$$

$$\mathrm{sinch}_\alpha(x) = \frac{\alpha \sin(\pi x)}{\sinh(\alpha \pi x)}$$
$$= \mathrm{sinc(x)}\left[(\alpha \pi x)\,\mathrm{cosech}(\alpha \pi x)\right]$$

Cardinal B-Splines are **NOT** better*
than Windowed Sincs

Because

They **ARE** windowed sincs!

Proof rather lengthy……20 pages of long hand algebra

CBS versus Sinch Complexity

- T. Briand (2018) detailed implementation of **CBS**, n=5, needs ~22 multiplications per interpolation (pixel)

  (Includes bilateral recursive filter x5)

  2D → **484** ops per pixel (or **44** ops x-y separable warp)

- **Sinch** order n=5 needs ~22 multiplications per interpolation (pixel)

  2D → **484** ops per pixel (or **44** ops x-y separable warp)

- **FFT** based flops $2 \times 5 \times \log_2 N$ per pixel..

  2D → $2 \times 5 \times \log_2 N$ ops per pixel =**280** (for 4kx4k image)

Who Doesn't Care?

Most Image Manipulation Software Resampling Options.
Some examples:

Photoshop	NN, Linear, Cubic
Paint Shop Pro	NN, Linear, Cubic, Wtd Av, Smart (picks best) , AI (picks best)
Matlab	NN, Linear, Cubic, Pchip, Modified Akima, Spline
IDL	Linear, Cubic (not in 3D)
ImageJ	NN, Linear, Cubic
GIMP	NN, Linear, Cubic, NoHalo (enlarge), LoHalo (shrink)
Irfanview	NN, Linear, Cubic, Bell, Mitchell, **B-spline (not CBS!)**, Hermite, Lanczos
Mathematica	Lanczos
Affinity	NN, Linear, Cubic, Lanczos 3 (separable or non-separable)
NIH MIPAV	NN, Linear, Cubic, **Spline (not CBS!)**, Lagrange, Gaussian, WinSinc

Central Premiss 4

Magic Kernel

- **globally most implemented resampling/resizing method**

... Facebook (2013) and Instagram* (2011)

Identical to

Cardinal B-Splines of

Aldroubi, Unser, Eden 1992

https://johncostella.com/magic/

* approx 10^12 image uploads per annum, implies N=? server-side resizing ops

Fourier Warping Algorithm

(4D complexity for 2D image
2D complexity for 1D signal)
i.e.order N^2 for N pixels

for non-affine warps

https://patents.google.com/patent/AU2005225121A1/en?q=(larkin+signal+warping)&assignee=canon&oq=larkin+signal+warping+canon&page=1

Interpolation Computational Complexities

- ChatGPT, ~3x10^14=

  300 Teraflop (typical)

- Affine Fourier Warp 2kx2k 3-shear =1.2x10^10= 0.01 Teraflop

- 4D Warp Algorithm 2kx2k, (4x10^6)^2=16 Teraflop

Non Affine Warps

- *Sinch* n=5 (2kx2k image), (19^2)x16x10^6=6x10^10=

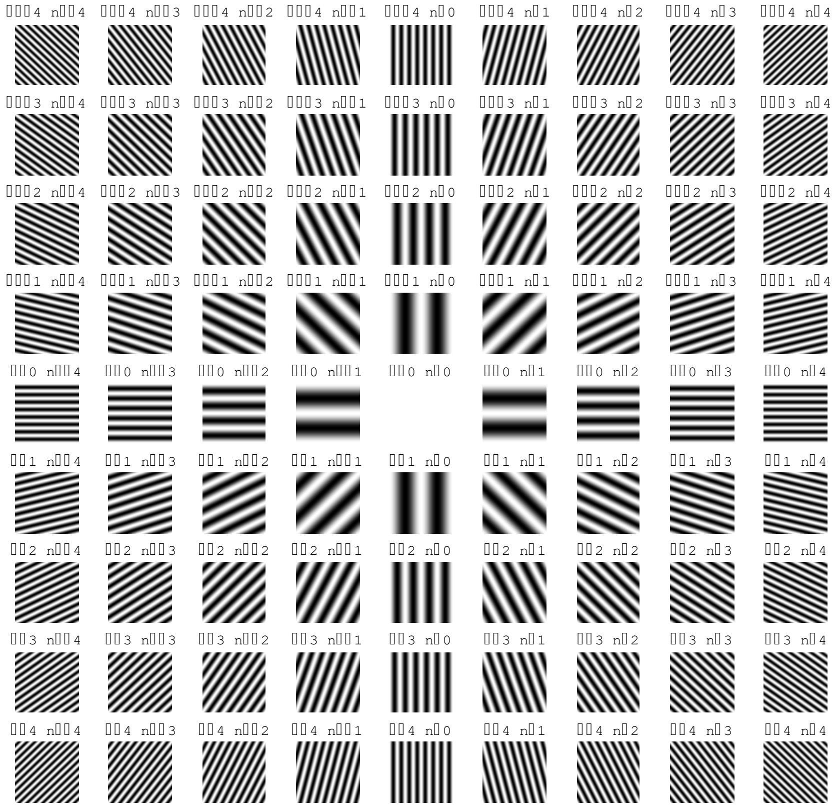
  0.06 Teraflop

- CBS n=5,~0.06 Teraflop

- Video, 1 Teraflop/s

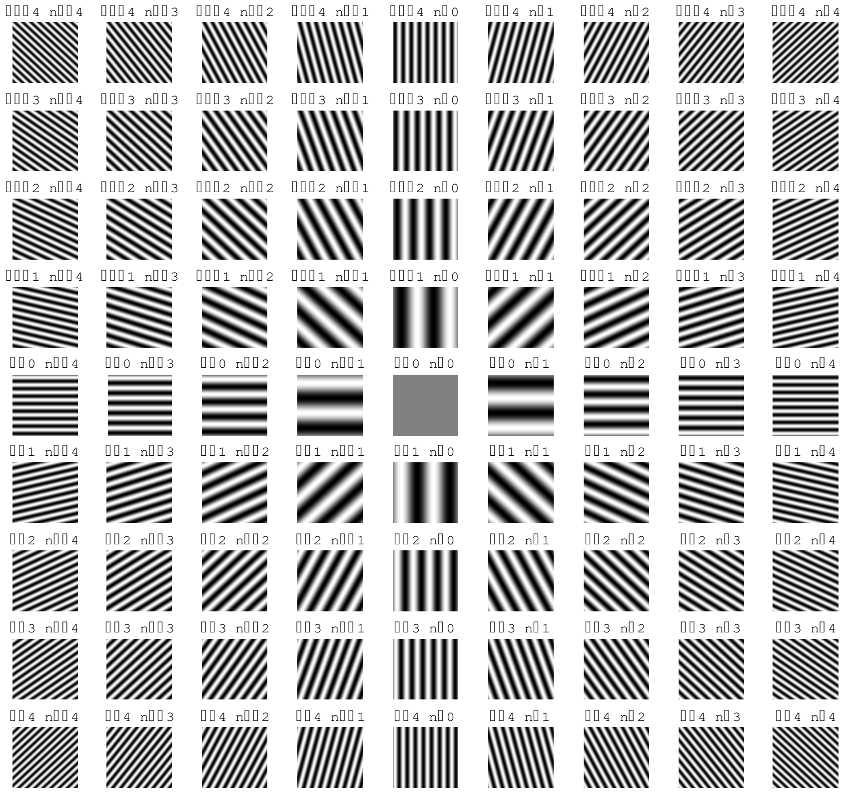
2D Fourier Basis Functions: 4D

cosine



2D array of 2D sub-images

sine



separately warp each sub-image

Real Function $\leftrightarrow$ Hermitian FT

- Only need half-plane of Fourier coeffs (*symmetry)
- Namely...Hilbert Trnsfrm
- Analytic signal +DC

$$f(x,y) \underset{}{\overset{FT}{\rightleftharpoons}} F(u,v)$$

$$F(u,v) = \left|F(u,v)\right| \exp\left[i\psi(u,v)\right]$$

$$g(x,y) \underset{}{\overset{FT}{\rightleftharpoons}} \mathrm{step}(x).F(u,v)$$

2D DFTs...centred-symm

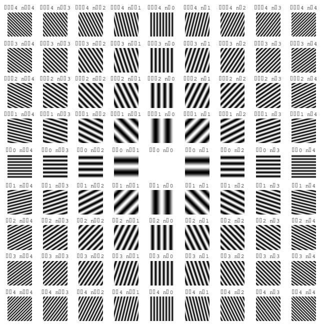
$$F(k,l) = \frac{1}{4MN} \sum_{n=-N}^{N-1} \sum_{m=-M}^{M-1} f(m,n) \exp\left[-2\pi i(mk+nl)\right]$$

$$f(m,n) = \frac{1}{4MN} \sum_{l=-N}^{N-1} \sum_{k=-M}^{M-1} F(k,l) \exp\left[+2\pi i(mk+nl)\right]$$

4D

$$g(m,n,k,l) = \frac{F(k,l)}{4MN}\exp\left[+2\pi i(mk+nl)\right]$$

$$f(m,n) = \sum_{l=-N}^{N-1}\sum_{k=-M}^{M-1} g(m,n,k,l)$$



Warp Fourier Bases

Now warp spatial coordinates: $(x, y) \rightarrow (x_w, y_w)$

$$m \rightarrow m_w = m + m_\varepsilon$$

$$n \rightarrow n_w = n + n_\varepsilon$$

$$f(m_w, n_w) = \frac{1}{4MN} \sum_{l=-N}^{N-1} \sum_{k=-M}^{M-1} F(k,l) \exp\left[+2\pi i (m_w k + n_w l)\right]$$

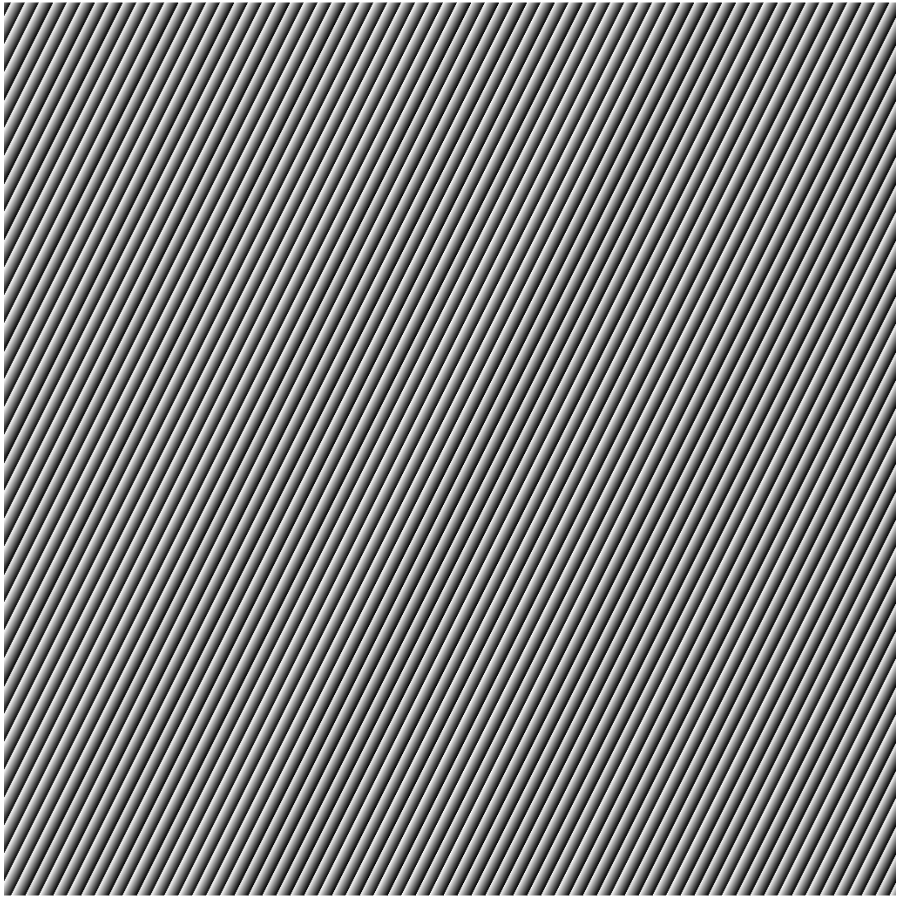
$$= \frac{1}{4MN} \sum_{l=-N}^{N-1} \sum_{k=-M}^{M-1} \left\{F(k,l) \exp\left[+2\pi i (mk + nl)\right]\right\} \exp\left[+2\pi i (m_\varepsilon k + n_\varepsilon l)\right]$$

Hence a warp is a complex multiplication of EACH 2D component (phase-shift fringes)

Pre-Warp Phase sub-image

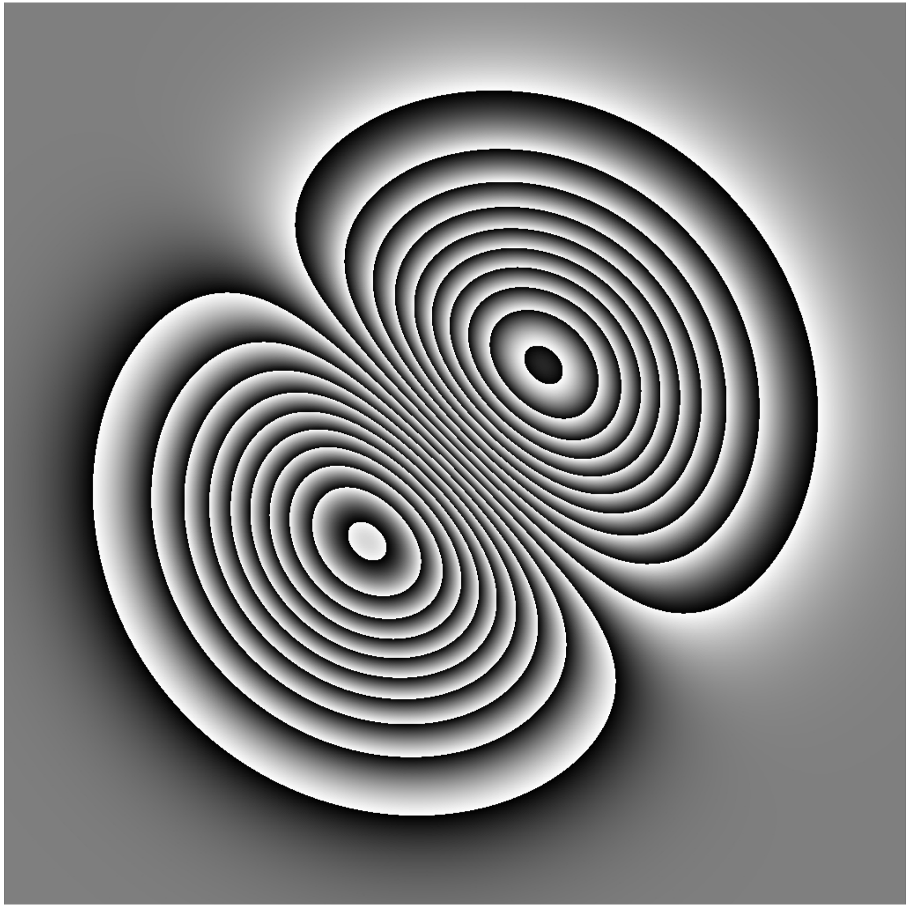
Combine cosine and sine bases into complex analytic pure phase fringe

$$\cos\left[2\pi i\left(mk+nl\right)\right]+i\sin\left[2\pi i\left(mk+nl\right)\right]$$
$$=\exp\left[+2\pi i\left(mk+nl\right)\right]$$



Warp Phase sub-image

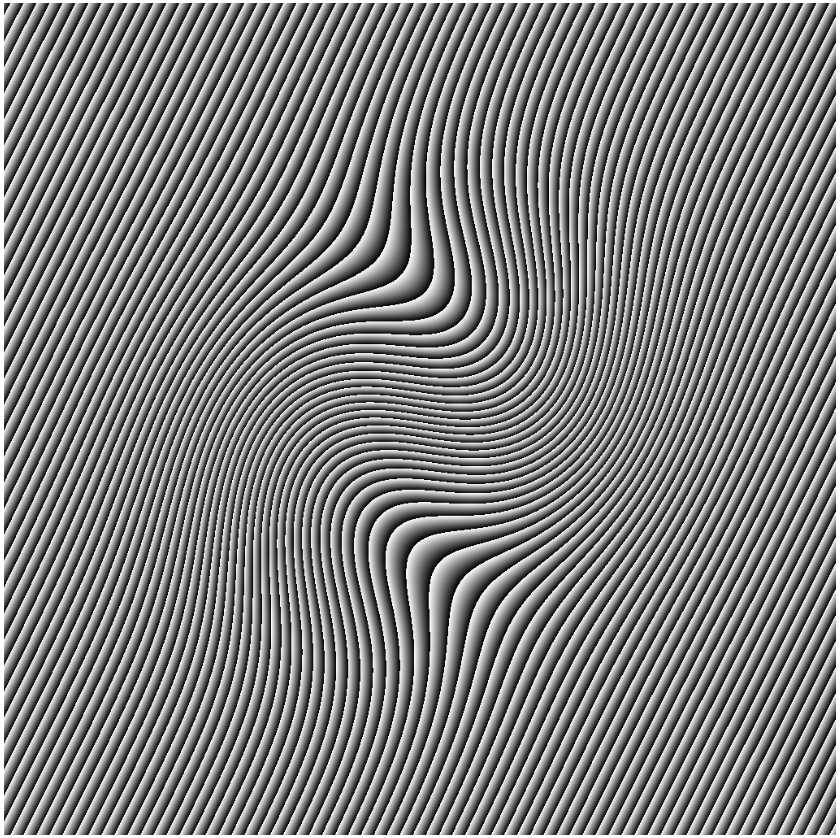
$$\exp\left[+2\pi i\left(m_{\varepsilon}k+n_{\varepsilon}l\right)\right]$$



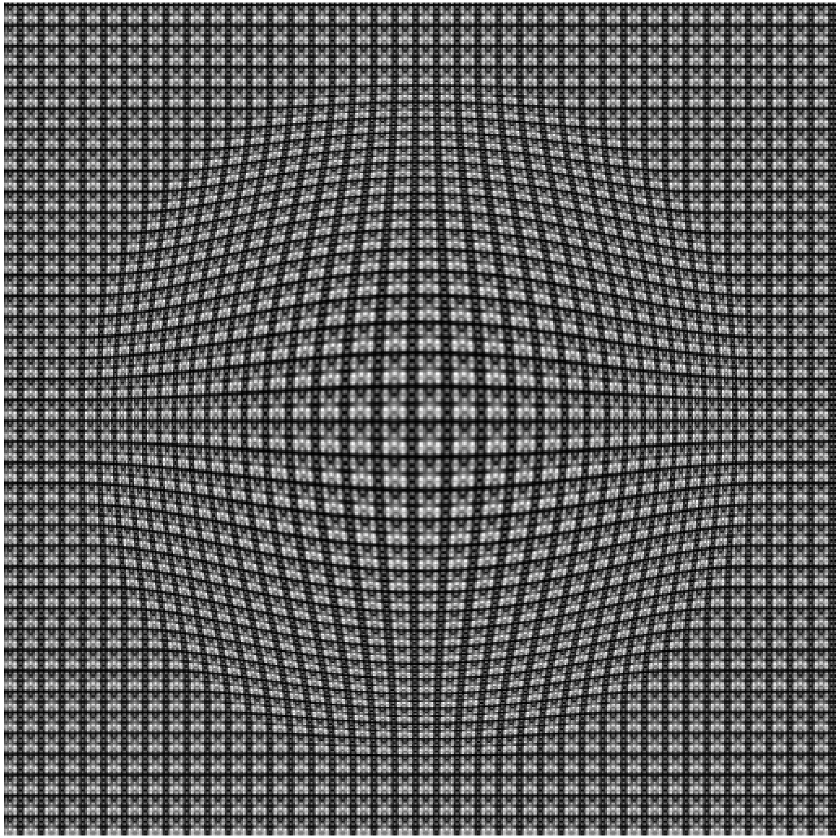
Resultant Warped spatial image

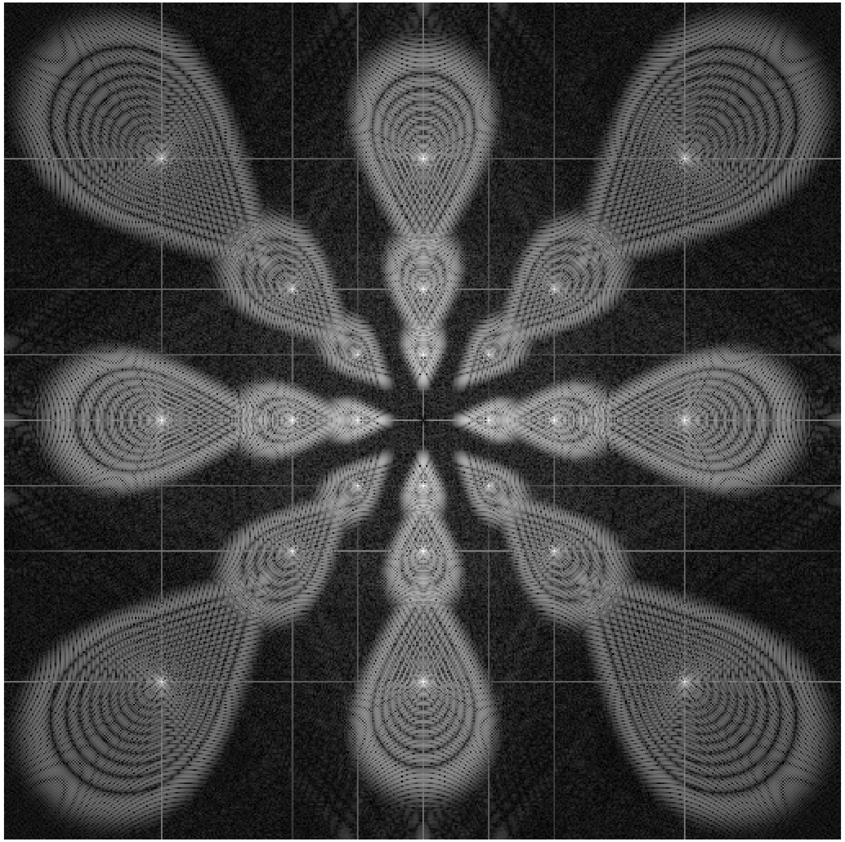
$$g\left(m_w, n_w, k, l\right) = g\left(m, n, k, l\right) \exp\left[+2\pi i\left(m_\varepsilon k + n_\varepsilon l\right)\right]$$

Repeat this for each fringe sub-image

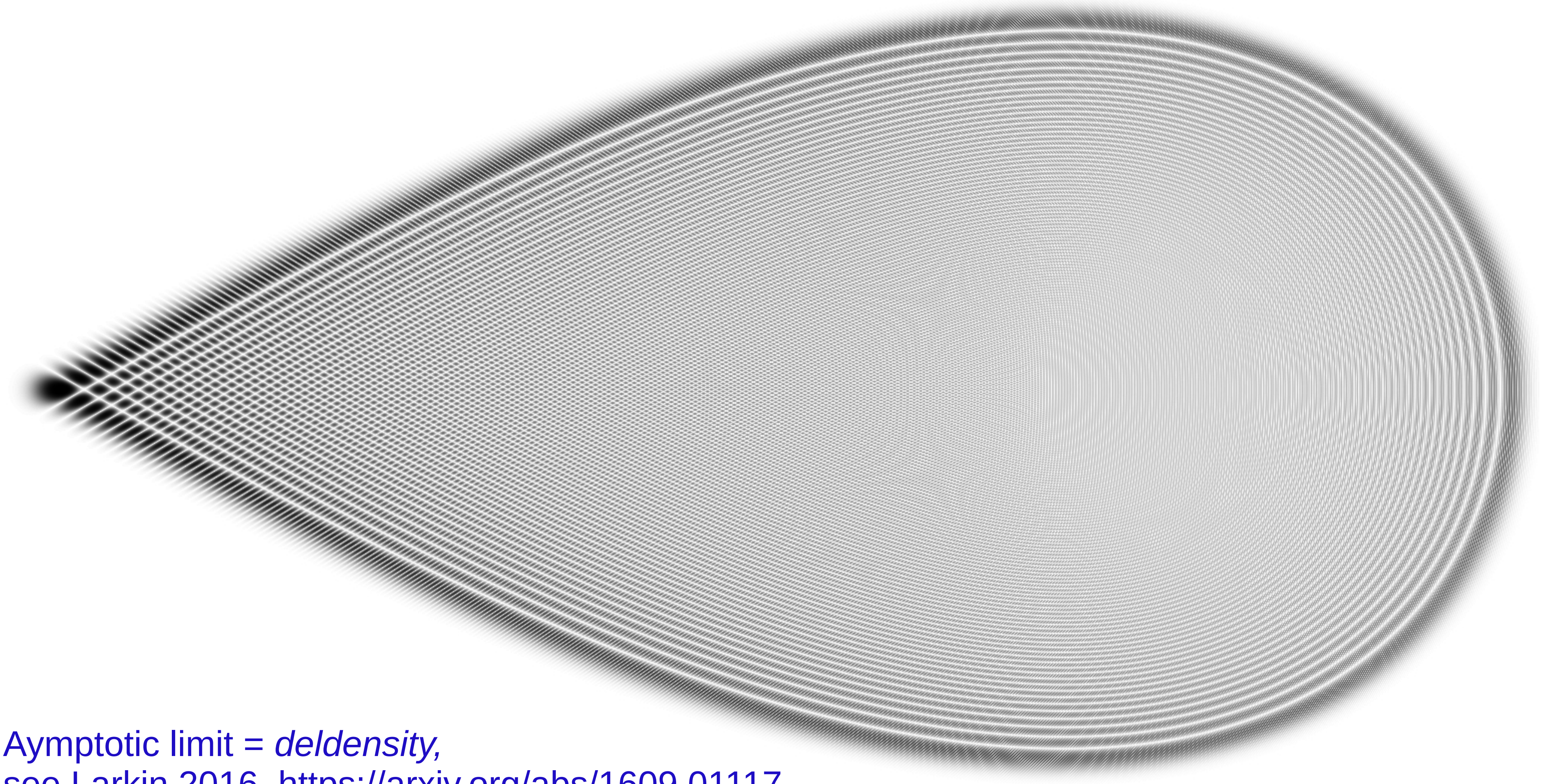


Warping: effect on (sparse) spectrum



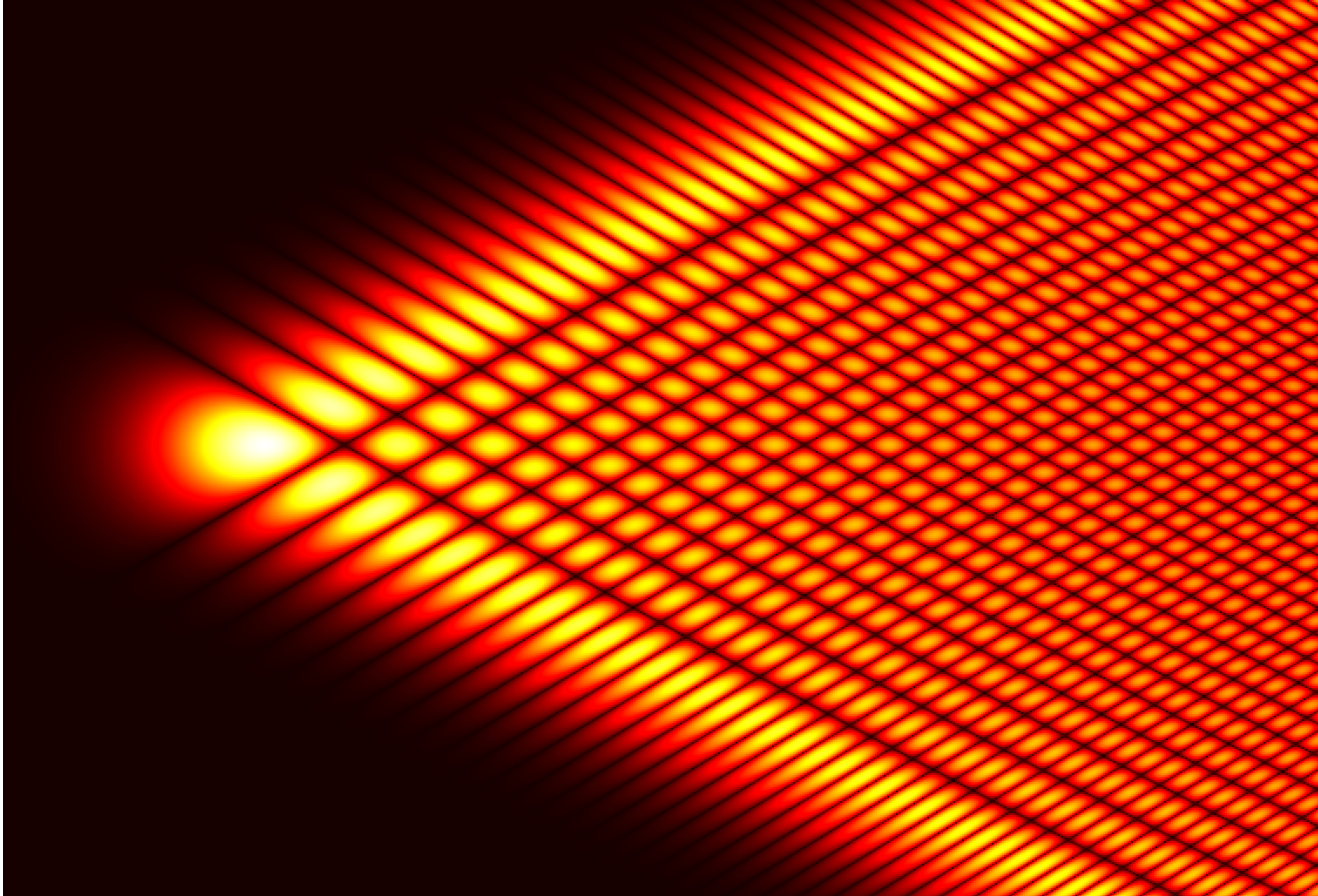


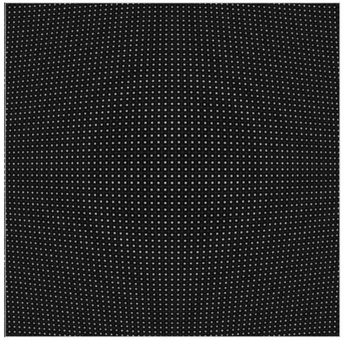
Spectral spread tractable by this approach….band-limit spreads/aliases



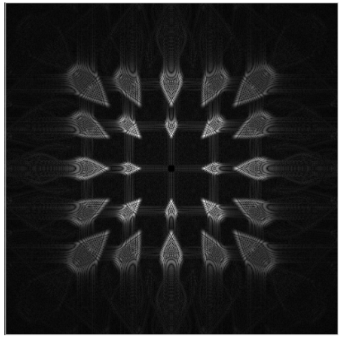
Aymptotic limit = *deldensity,*
see Larkin 2016, https://arxiv.org/abs/1609.01117

Compare Nijboer Born & Wolf diffraction from aberrations ...

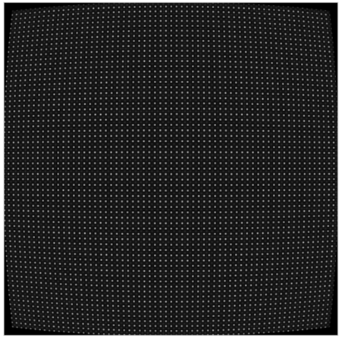




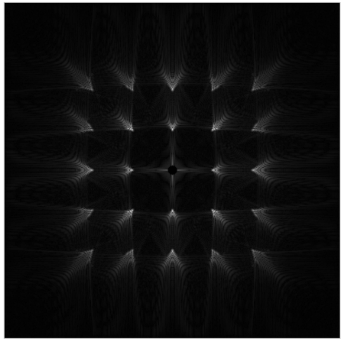
Warp



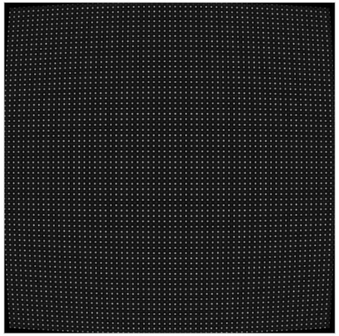
Fourier Spectrum



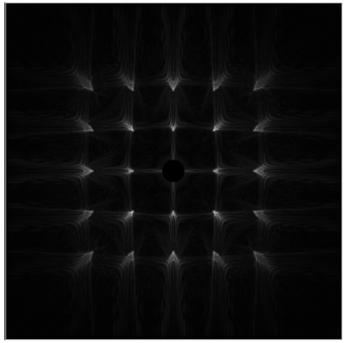
Barrel 20



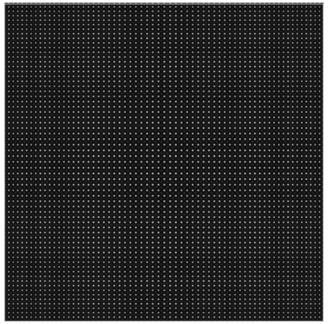
Fourier Spectrum



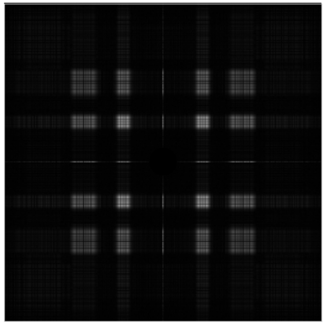
Fisheye 40



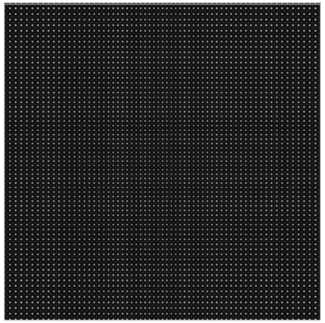
Fourier Spectrum



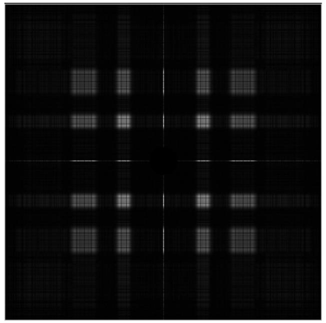
Punch 20



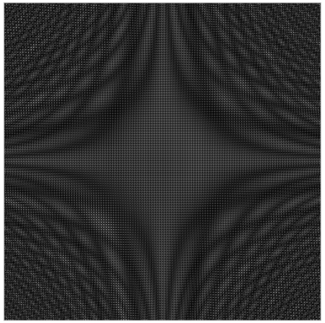
Fourier Spectrum



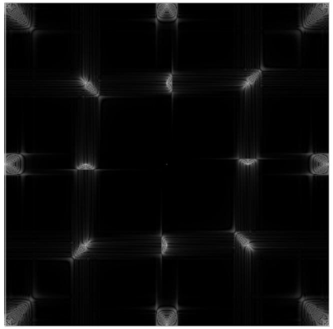
Pinch 20



Fourier Spectrum



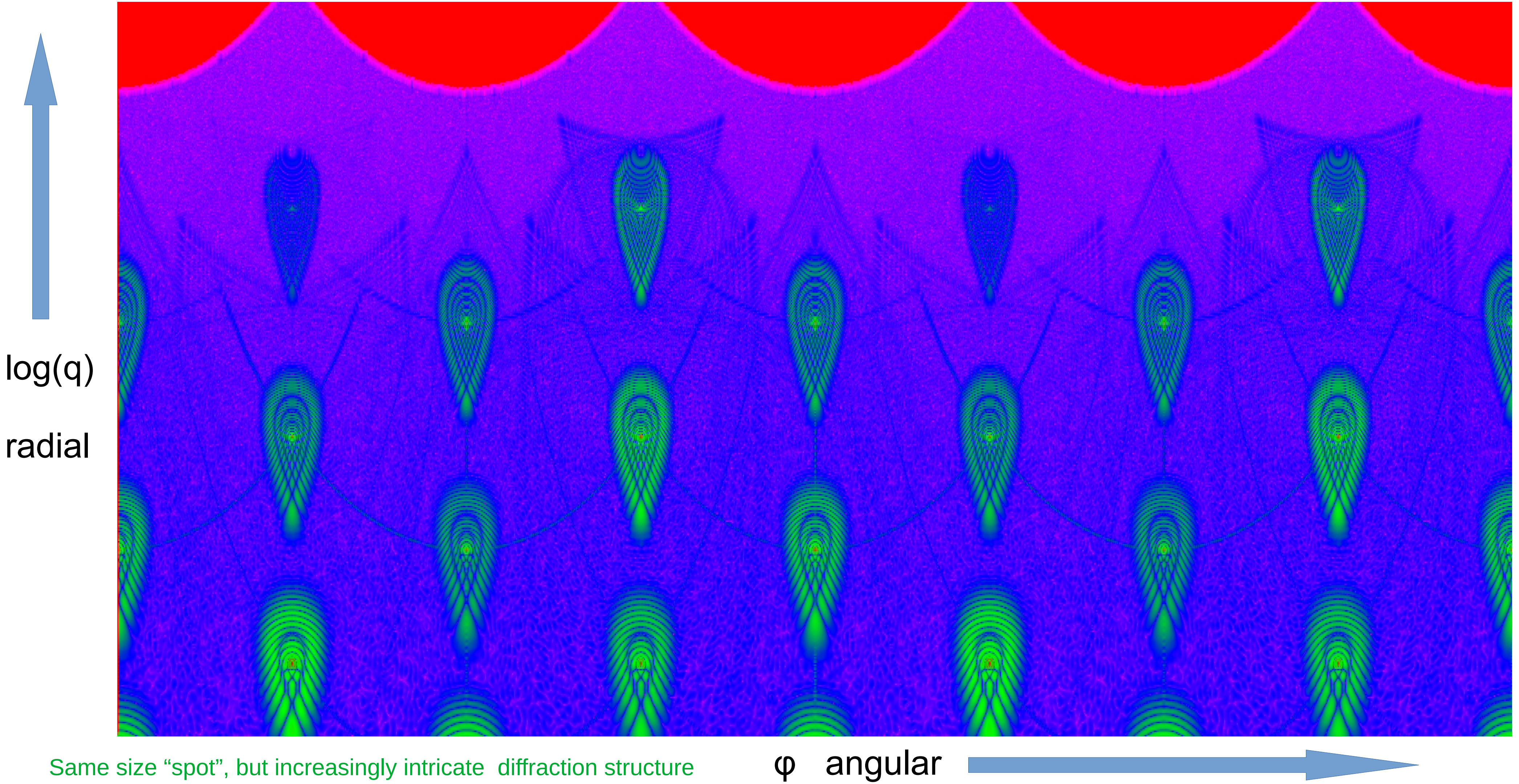
Hyperbolic warp (below)



Fourier Spectrum

0.5+cos(pi*(1+x*y/1000000)*x/2)+cos(pi*(1+x*y/1000000)*y/2)++

Log-Polar Fourier Transform



Same size “spot”, but increasingly intricate diffraction structure

Bracewell correspondence...

The Oxometrical Society

- Measurement of Bullshit

- In 1944 the Oxometrical Society of Sydney University awarded the degree of Doctor to Ern Malley for having shown himself a notable producer of oxoplasm

Date: **Thu, 4 Jun 1998** 09:52:35 -0700 (PDT)
To: Kieran Larkin <kieran@research.canon.com.au>
From: "R.N. Bracewell" <bracewell@NOVA.Stanford.EDU>
Subject: Re: Sampling and Interpolation in Two Dimensions

Dear Kieran,

Thank you for the kind remarks. Not many such letters arrive.

Suppose you want to interpolate f = [1 4 6 4 1]. Matlab tels me that fft(f) is let's say [a b c d e]. Then ifft([a b c 0 0 0 0 0 d e]) is

1 2.31 4 5.44 6 5.44 4 2.31 1 .52

How does this compare?

I don't know anything about chirp-z transform. Could you fax me your Optics Comm paper to (650) 723-3545?

I'm off to Italy tomorrow until July 16. I have just about finished a 3rd edition of FTA. If I were to insert a short section on the chirp-z transform in Chap. 12, what should I say?

Best wishes,

Ron Bracewell

Bracewell correspondence

Fourier shift theorem

Bracewell correspondence

exoplanet detection interferometry

>Date: **Wed, 7 Oct 1998** 15:10:46 -0700 (PDT)
>X-Sender: bracewel@nova.stanford.edu
>Mime-Version: 1.0
>To: colin@Physics.usyd.edu.au (Colin Sheppard)
>From: "R.N. Bracewell" <bracewell@nova.stanford.edu>
>Subject: Visit
>
>Dear Colin,
>
> Sorry to miss you in Sydney. I did have the chance to talk to
>Kieran Larkin, who is a most interesting fellow. He gave me some material
>that I plan to absorb and had sent some stuff earlier.
>
> For a quaint development in interferometry that would have been
>unpredictable twenty years ago see Hinz et al. Nature September 17, also
>Angel and Woolf Ap. J. January 1997.
>
> Regards,
>
> Ron

Bracewell correspondence: FractFT

Ronald N. Bracewell
L.M. Terman Professor of Electrical Engineering Emeritus
Stanford University, Stanford, CA 94305-4055

November 23, 1998

To: Kieran Larkin
011 612 9805-2929

Dear Kieran,

Your articles in the *Australian Optical Society NEWS* were very enjoyable and helpful to me in thinking about how to present the Frac FT in my 3rd ed. Unfortunately, it will not be possible to carry over your level of humour. I am trying to write the most compact section that can be fitted into Chap. 12 following the Hilbert transform. Since the dry mathematical definition looks a bit forbidding I am starting with a physical approach. In addition I am trying to make the definition itself a little simpler by restricting to $0 < a < 1$ so as to get rid of the perplexing $\text{sgn}(\sin\phi)$. I am not sure that I have got this right.

THE END...

QUESTIONS...

ANSWERS

Application (animation)

Warping/de-warping image of test patterns

For

Optical systems *ultraprecision* image quality measurement
(see numerous Canon patents)