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The three-dimensional transfer function and phase space mappings

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Abstract: For paraxial wave-fields in two dimensions, it is known that the defocused OTF is given by a section through the ambiguity function. This is no longer true for high aperture fields. It is shown that there is a simple relationship, even for high aperture fields, between the two-dimensional (one transverse and one longitudinal) generalized OTF and the spectral correlation function. The connection with the Wigner distribution function is also discussed. The generalization to three-dimensional wavefields is also considered.

Key words: Transforms, Wigner distribution function – ambiguity function – OTF, phase retrieval – propagation – diffraction – phase space representations

1. Introduction

We recently presented a direct method for phase retrieval from intensity measurements of cylindrical wavefields [1]. In this method the two-dimensional (2-D) intensity distribution in the xz plane was Fourier transformed to give a 2-D (one transverse and one longitudinal) generalized optical transfer function (OTF), that after coordinate transformation could be separated to yield the complex amplitude of the illuminating wave (the 2-D generalized pupil function) [2]. Our method applies for scalar waves with high angles of convergence, for which the 2-D generalized OTF has been presented previously [3, 4]. The generalized OTF has been extended to 3-D [5–8], and the principle of the 3-D generalized pupil function can also be developed for the vectorial, electromagnetic, case [9].

Subsequently, the similarity of our approach for phase retrieval with methods based on tomography of the Wigner distribution function [10, 11] or the ambiguity function [12] has come to our attention. These latter papers treat the phase retrieval problem for 2-D or 3-D wavefields using a paraxial model, in which propagation along the axis results in a shear of the Wigner or ambiguity functions. The aim of this paper is to present explicit expressions for the connection between these 2-D Wigner and ambiguity functions and the 2-D generalized OTF (for 2-D wave-fields), which can then be extended to the case of 4-D Wigner and ambiguity functions, together with 3-D generalized OTFs (for 3-D wave-fields). It is interesting to note that although the concept of the generalized OTF dates back to the mid-sixties, and the Wigner and ambiguity functions were applied to optics to the seventies, to our knowledge no papers have discussed the connection between these concepts.

2. The ambiguity function and the 2-D OTF for 2-D wave-fields in the paraxial approximation

The defocused OTF of a convergent wave-field can be expressed directly in terms of the ambiguity function [13]. For a wave-field represented by a 1-D pupil, the defocused pupil function can be written

$$P(m;u) = P_0(m)\exp(\mathrm{i}kW_{20}m^2) = P_0(m)\exp\left(\frac{1}{2}\mathrm{i}um^2\right) \quad (1)$$

where $P_0(m)$ is the in-focus pupil function and m is a normalized spatial frequency so that

$$P_0(m) = 0, \quad m\,|>|\,1\,. \quad (2)$$

Also, W_{20} is the defocus coefficient, $k = 2\pi/\lambda$, and u is a normalized axial displacement

$$u = 4kz\sin^2\frac{\alpha}{2} \quad (3)$$

where α is the angular aperture of the lens. Defocus is represented by a parabolic phase factor in eq. 1. Then the defocused OTF is given by

$$C(m;u) = \frac{\int P(m'+m/2)\,P^*(m'+m/2)\,\mathrm{d}m'}{\int|P(m')|^2\,\mathrm{d}m'} = \frac{1}{E}\int P_0(m'+m/2)\,P_0^*(m'-m/2)\exp(\mathrm{i}umm')\,\mathrm{d}m' \quad (4)$$

where E is the energy in the beam. All integrals are taken to be evaluated from minus to plus infinity, with the pupil

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function taken as zero outside of its aperture. The ambiguity function is defined as [14]

$$A(m,x)=\frac{1}{E}\int P_0(m'+m/2)\,P_0^*(m'-m/2)\cdot\exp(\mathrm{i}\,2\pi m'x)\,\mathrm{d}m'\,. \qquad (5)$$

Comparing eqs. 4 and 5, we thus have a connection between the defocused OTF and the ambiguity function [13], with the relationship

$$um=2\pi x\,. \qquad (6)$$

Papoulis [14] also considers the spectral correlation function, defined as

$$\gamma(m,m')=\frac{1}{E}\,P_0(m'+m/2)\,P^*(m'-m/2)\,, \qquad (7)$$

so that we then also have

$$C(m;u)=A\left(m,\frac{um}{2\pi}\right)=\int\gamma(m,m')\exp(\mathrm{i}\,umm')\,\mathrm{d}m'. \qquad (8)$$

Consider now the 2-D generalized pupil function, given by the 1-D Fourier transform (in the longitudinal direction) of the defocused pupil function,

$$\Pi(m,s)=P_0(m)\int\exp\left(\frac{1}{2}\,\mathrm{i}\,um^2\right)\exp(-\,\mathrm{i}\,us)\,\mathrm{d}u\,. \qquad (9)$$

The significance of the 2-D generalized pupil function is that its 2-D inverse Fourier transform gives the amplitude in the focal region. Then there are simple relationships between the 2-D generalized pupil function and the 1-D in-focus pupil function:

$$\Pi(m,s)=P_0(m)\,\delta(s-m^2/2), \qquad (10)$$

and also

$$P_0(m)=\int\Pi(m,s)\,\mathrm{d}s\,. \qquad (11)$$

In the same way, the 2-D generalized OTF is given by the Fourier transform of the defocused OTF [3]

$$C_2(m,s)=\int C(m;u)\exp(-\,\mathrm{i}\,us)\,\mathrm{d}u\,. \qquad (12)$$

The 2-D generalized OTF is given by the autocorrelation of the 2-D generalized pupil function. The significance of the 2-D generalized OTF is that its 2-D inverse Fourier transform gives the intensity in the focal region. Substituting eq. 8 in eq. 12, we then have

$$C_2(m,s)=\frac{2\pi}{|m|}\,\gamma\left(m,\frac{s}{m}\right) \qquad (13)$$

where

$$s/m=m', \qquad (14)$$

so that

$$\gamma(m,m')=\frac{|m|}{2\pi}\,C_2(m,mm')\,. \qquad (15)$$

Eqs. 13 and 15 represent a coordinate transformation and a rescaling. In fig. 1 we illustrate how parabolas in m, s space representing contours of constant $m'\pm m/2$ are mapped into straight lines in m, m' space.

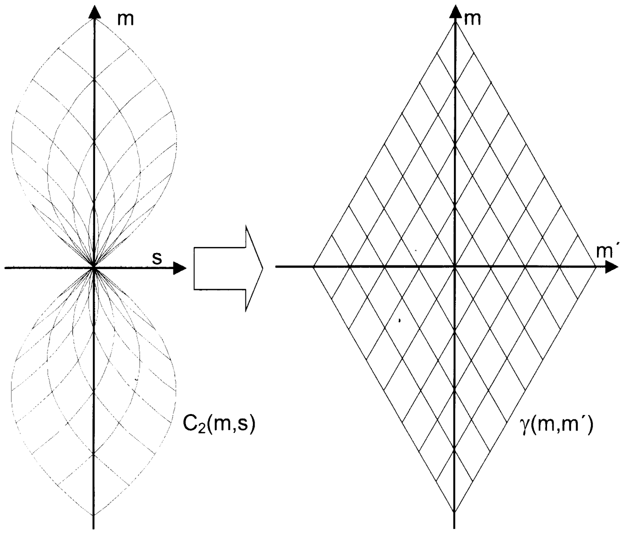

Fig. 1. The coordinate transformation between the 2-D generalized OTF in m, s and the spectral correlation function in m, m' space. Parabolas in m, s space respresentig contours of constant $m'\pm m/2$ are mapped into straight lines in m, m' space.

Consider as an example an aberration-free pupil of constant amplitude unity for $|m|<1$. The corresponding 2-D OTF and spectral correlation function are illustrated in fig. 2. The 2-D OTF exhibits a singularity at the origin, which is eliminated by the transformation in eq. 15.

In terms of the ambiguity function, inverting eq. 5

$$C_2(m,s)=\frac{2\pi}{|m|}\int A(m,x)\exp\left(-\frac{\mathrm{i}\,2\pi xs}{m}\right)\mathrm{d}x \qquad (16)$$

and by inverting eq. 12

$$A(m,x)=\frac{1}{2\pi}\int C_2(m,s)\exp\left(\frac{\mathrm{i}\,2\pi xs}{m}\right)\mathrm{d}s\,. \qquad (17)$$

Our phase retrieval method [1] is effectively based on determination of the spectral correlation function by Fourier transformation of the focal intensity to obtain the 2-D OTF, followed by a coordinate transformation or mapping. Eq. 15 succinctly represents the paraxial form of our phase retrieval algorithm. Because C_2 is the 2-D Fourier transform of the intensity distribution, and γ is essentially a 1-D remapping of C_2 we see that γ may be efficiently calculated by combining a 1-D Fourier operation and the 1-D remapping (interpolation) operation into one. The method is then equivalent to the (strictly paraxial) method of Tu and Tamura [15] of recovering the mutual intensity, using the AF with certain computational advantages over the alternative method of WDF tomography [10, 11].

Turning now to the Wigner distribution function [16]

$$W(m',x')=\int\gamma(m,m')\exp(\mathrm{i}\,2\pi mx')\,\mathrm{d}m \qquad (18)$$
$$=\iint A(m,x)\;\exp(\mathrm{i}\,2\pi(mx'-m'x))\,\mathrm{d}m\mathrm{d}x$$

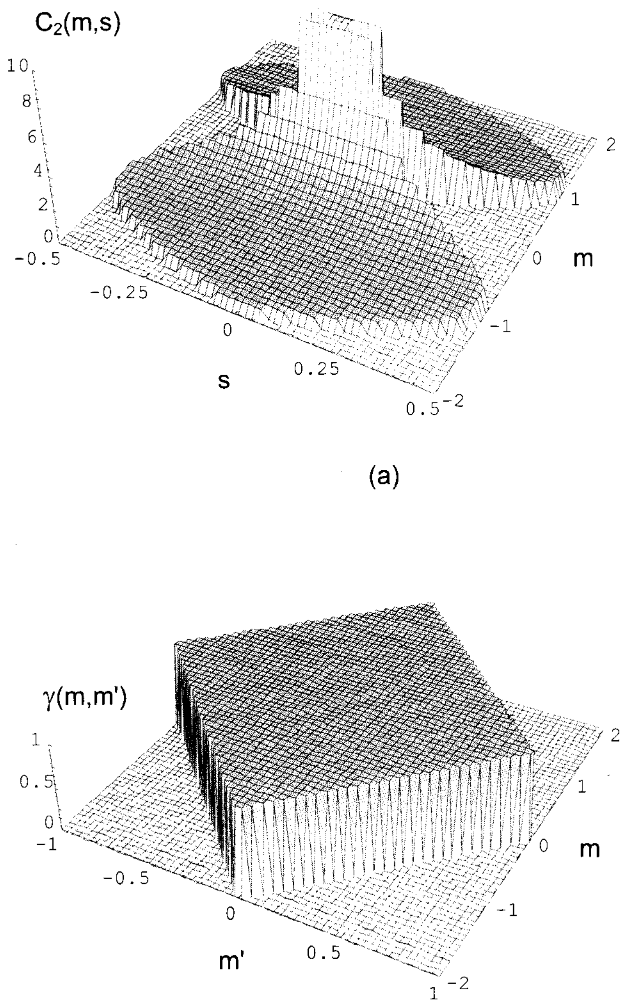

Fig. 2. The 2-D generalized OTF and the spectral correlation function for an unaberrated slit pupil.

we have the relationships

$$C_2(m,s) = \frac{2\pi}{|m|} \int W\left(\frac{s}{m'}, x'\right) \exp(-\mathrm{i}\,2\pi m x')\,\mathrm{d}x' \tag{19}$$

$$W(m',x') = \int \frac{|m|}{2\pi} C_2(m, mm') \exp(\mathrm{i}\,2\pi m x')\,\mathrm{d}m\,. \tag{20}$$

3. The 2-D OTF for a 2-D wave-field of high angle of convergence

We now consider the effects of a high aperture system. In this case defocus is represented by a spherical, rather than a parabolic, phase variation. Then if ϕ is the angle subtended with the axis by a point on the pupil corresponding to normalized spatial frequency m

$$m = \frac{\sin\theta}{\sin\alpha'} \tag{21}$$

and

$$P(m;u) = P_0(m)\exp(-\mathrm{i}kz(1-\cos\theta)), \tag{22}$$

so that, making the phase at the edge of the pupil the same as for the paraxial case,

$$P(m;u) = P_0(m) \cdot \exp\left(-\frac{\mathrm{i}u}{4\sin^2\frac{\alpha}{2}}\left(1-(1-m^2\sin^2\alpha)^{1/2}\right)\right). \tag{23}$$

We see from eqs. 4, 7 and 12 that a rescaling and coordinate mapping relationship between the 2-D generalized OTF and the spectral correlation function still occurs, but the factors are different from those given in eqs. 13 and 15 [17]. However, the relationship between the ambiguity function and the defocused OTF is no longer simple. The fractional Fourier transform representation also fails outside of the paraxial regime. Our conclusion is that the ambiguity function, and therefore also the Wigner distribution function (as conventionally defined), are not useful representations for high aperture fields. This conclusion was also reached by Wolf et al. [18], when they considered 4-D Wigner distribution functions for 2-D wave-fields.

Note, however, that defining 4-D forms of the Wigner [18] and ambiguity functions lead to a simple relationship. From eqs. 4, 9, and 12, the 2-D generalized OTF is given in terms of the autocorrelation of the 2-D generalized pupil function

$$C_2(m,s) = \frac{1}{E}\frac{1}{4\pi^2}\iint \Pi(m'+m/2, s'+s/2) \cdot \Pi^*(m'-m/2, s'-s/2)\,\mathrm{d}m'\mathrm{d}s'. \tag{24}$$

Defining the 4-D ambiguity function

$$A(m,s;x,z) = \frac{1}{E}\iint \Pi(m'+m/2, s'+s/2) \cdot \Pi^*(m'-m/2, s'-s/2) \cdot \exp(\mathrm{i}\,2\pi(m'x+s'z))\,\mathrm{d}m'\mathrm{d}s' \tag{25}$$

we then see that

$$C_2(m,s) = \frac{1}{4\pi^2} A(m,s;0,0)\,. \tag{26}$$

4. Three-dimensional wave-fields

The treatment of section 2 can be extended to the case of 2-D pupils. Then

$$P(m,n;u) = P_0(m,n)\exp\left(\frac{1}{2}\mathrm{i}u(m^2+n^2)\right). \tag{27}$$

The spectral correlation function is

$$\gamma(m,m';n,n') = \frac{1}{E} P_0(m'+m/2, n'+n/2) \cdot P_0^*(m'-m/2, n'-n/2). \tag{28}$$

The defocused OTF is

$$C(m,n;u) = \iint \gamma(m,m';n,n') \cdot \exp(\mathrm{i}u(mm'+nn'))\,\mathrm{d}m'\mathrm{d}n' \tag{29}$$

The ambiguity function is

$$A(m, x; n, y) = \iint \gamma(m, m'; n, n') \cdot \exp(\mathrm{i}2\pi(m'x + n'y))\, \mathrm{d}m'\mathrm{d}n'. \quad (30)$$

Comparing eqs. 29, and 30 we have

$$C(m, n; u) = A\left(m, \frac{um}{2\pi}; n, \frac{un}{2\pi}\right). \quad (31)$$

Thus knowledge of the complete defocused OTF determines only a 3-D section through the 4-D ambiguity function. Alternatively, this can be interpreted as saying that determination of the 3-D wave-field requires only specification of a 3-D section through the 4-D ambiguity function.

The 3-D generalized OTF is given by the Fourier transform of $C(m, n; u)$ with respect to u, giving

$$C_3(m, n, s) = 2\pi \iint \gamma(m, m'; n, n') \cdot \delta((s - (mm' + nn'))\, \mathrm{d}m'\mathrm{d}n'. \quad (32)$$

Thus the 3-D generalized OTF is given by a line integral on a straight path across the region of overlap of two displaced pupils, as originally pointed out by Frieden [5]. The 3-D generalized OTF can also be expressed in terms of the ambiguity function

$$C_3(m, n, s) = \int A\left(m, \frac{um}{2\pi}; n, \frac{un}{2\pi}\right) \exp(-\mathrm{i}us)\, \mathrm{d}u. \quad (33)$$

5. Conclusions

For 2-D paraxial wave-fields a simple relationship exists between the 2-D generalized OTF and the spectral distribution function, representing a coordinate transformation and a rescaling. The defocused OTF is given by a section through the ambiguity function. A Fourier transform relationship exists between the 2-D generalized OTF and the ambiguity function.

For 2-D wave-fields of high aperture, the relationship between the 2-D generalized OTF and the spectral correlation function is still valid, but the relationship between the ambiguity function and the defocused OTF is no longer simple. The conventionally defined ambiguity function is not a useful representation for high aperture fields.

For 3-D paraxial wave-fields, knowledge of the complete defocused OTF determines only a 3-D section through the 4-D ambiguity function. The 3-D generalized OTF is given by a line integral on a straight path across the region of overlap of two displaced pupils. The 3-D generalized OTF can also be expressed as a Fourier transform of the ambiguity function.

Relationships can be developed between the 4-D (and 6-D) ambiguity functions for 2-D (3-D) high aperture wave-fields and the generalized OTF.

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